Strong and weak error estimates for the solutions of elliptic partial differential equations with random coefficients

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Groundwater steady flow

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$$\mathrm{div}(a(x)\nabla p(x)) = 0 \quad \forall x \in D \subset \mathbb{R}^d$$
$$+\text{boundary conditions.}$$

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  where g is a mean-free **gaussian field**, its law is uniquely determined by the covariance function supposed to be homogeneous, i.e.

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- Goal: compute the law of $p : \Omega \to H^1(D)$ i.e. the $\mathbb{E}[\varphi(p)](x)$
  in practice, we compute the first two moments, i.e. the functions $\mathbb{E}[p](x)$ and $\mathbb{E}[p^2](x)$.

Equation and assumptions

- D a bounded open $\mathcal{C}^2$ domain of $\mathbb{R}^d$, $(\Omega, \mathcal{F}, P)$ a probability space
- $a : \Omega \times \bar{D} \to \mathbb{R}$ a lognormal homogeneous random field $a(\omega, x) = e^{g(\omega,x)}$ where g is a gaussian homogeneous mean-free random field with $\mathrm{cov}[g](x,y) = k(\|x-y\|)$, $k \in \mathcal{C}^{0,1}(\mathbb{R})$
- We look for $u : \Omega \times D \to \mathbb{R}$ such that for almost every ω

$$\left\{ \begin{array}{rcll} -\mathrm{div}(a(\omega,.)\nabla u(\omega,.)) & = & f(x) & \text{on } D \\ u(\omega,.) & = & 0 & \text{on } \partial D. \end{array} \right.$$

Proposition

For almost all ω, $a(\omega, .) \in \mathcal{C}^{0,\alpha}(\bar{D})$ *for any* $\alpha < \frac{1}{2}$.

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Key point of the proof:

Theorem (Kolmogorov)

Let $X(\omega, x) \ : \ \Omega \times D \subset \mathbb{R}^d \to \mathbb{R}^n$ *be a stochastic process such that there exists a constants* C, $p > 1$ *and* $\varepsilon > 0$ *such that for any* $x, y \in D$ *we have*

$$\mathbb{E}[\|X(\omega,x) - X(\omega,y)\|^p] \leq C\|x-y\|^{d+\varepsilon},$$

then for almost all ω, $X(\omega,.) \in \mathcal{C}^{0,\beta}(\bar{D})$ *for any* $\beta < \frac{\epsilon}{p}$.

Proposition

We can then define for almost all ω*:*

$a_{min}(\omega) = \min_{x\in\bar{D}} a(\omega, x)$ *and* $a_{max}(\omega) = \max_{x\in\bar{D}} a(\omega, x)$.

Then $\frac{1}{a_{min}(\omega)} \in L^p(\Omega)$ *and* $a_{max}(\omega) \in L^p(\Omega)\ \forall p > 0$.

Key point of the proof:

Theorem (Fernique)

If E *is a separable Banach space and* X *a mean-free gaussian random variable with values in* E*, then there exists* $\alpha > 0$ *such that*

$$\mathbb{E}[e^{\alpha\|X\|_E^2}] < \infty,$$

in particular, for any $p > 0$, we have $\mathbb{E}[e^{p\|X\|_E}] < \infty$.

Proposition

The equation

$$\begin{cases} -\mathrm{div}(a(\omega,.)\nabla u(\omega,.)) & = & f(x) & \text{on } D \\ u(\omega,.) & = & 0 & \text{on } \partial D. \end{cases}$$

admits a unique solution $u \in L^p(\Omega, H_0^1(D))$, $\forall p > 0$.

Proof

For a.e. ω, the equation admits a unique solution $u(\omega,.) \in H_0^1(D)$ with

$$\|u(\omega,x)\|_{H_0^1(D)} \leq \frac{C_D}{a_{min}(\omega)}\|f\|_{L^2(D)}.$$

Therefore

$$\|u\|_{L^p(\Omega,H_0^1(D))} \leq C_D\|f\|_{L^2(D)}\left\|\frac{1}{a_{min}}\right\|_{L^p(\Omega)}.$$

Approximation of a

We approximate the random field $a(\omega, x)$ by a function of x and of N random variables, i.e. in a finite dimensional stochastic space:

$$a(\omega, x) \rightsquigarrow \tilde{a}(Y_1(\omega), ..., Y_N(\omega), x).$$

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- It is the first and fundamental step of several numerical methods: some perturbation-type methods, and mostly the spectral stochastic methods: stochastic collocation methods and stochastic galerkin methods.

Basic introduction to the stochastic collocation method

- By Doob Dynkin lemma, the solution u of the equation: for a.e. ω

$$\begin{cases} -\mathrm{div_x}(\tilde{a}(Y_1(\omega),...,Y_N(\omega),x)\nabla_x \tilde{u}(\omega,x)) & = & f(x) & \text{on } D \\ \tilde{u}(\omega,x) & = & 0 & \text{on } \partial D \end{cases}$$

  can be expressed as a function of $Y(\omega) = (Y_1(\omega),...,Y_N(\omega))$ and x, i.e. $\tilde{u}(\omega,x) = \bar{u}(Y_1(\omega),...,Y_N(\omega),x)$.

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- The equation solved by $\bar{u}$ is then: for any $y = (y_1, ...y_N) \in \mathbb{R}^N$

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- If Y admits the density $\rho : \mathbb{R}^N \to \mathbb{R}$, then $\mathbb{E}[\varphi(\tilde{u}(\omega, x))] = \int_{\mathbb{R}^N} \varphi(\bar{u}(y, x))\rho(y)dy \ \in H^1(D)$ can be approximated with quadrature rules in $\mathbb{R}^N$, i.e. by $\sum_{i=1}^{p_N} \lambda_i \varphi(\bar{u}(y_i, x))$.
  $\rightsquigarrow$ the computational cost increases drastically with N.

Approximation of a

How to approximate a?

- Let $(\lambda_n, b_n)_{n\in\mathbb{N}}$ be the eigenpairs of the Hilbert-Schmidt operator:

$$f \in L^2(D) \quad \longmapsto \quad \left(x \mapsto \int_D cov[g](x,y)f(y)dy\right) \in L^2(D)$$

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  The $(b_n)_n$ and the $(Y_n)_n$ are orthonormal.
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- Then the Karhunen-Loève expansion of g is:

$$g(\omega, x) \overset{L^2(\Omega\times D)}{=} \sum_{n=1}^{+\infty} \sqrt{\lambda_n} b_n(x) Y_n(\omega)$$

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- We approximate then g by the truncated expansion at order N,
  $g_N(\omega, x) = \sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega)$.

- We define the approximation a_N of a:

$$a_N(\omega, x) = e^{g_N(\omega,x)} = e^{\sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x) Y_n(\omega)}.$$

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- We define the approximation u_N of u as the solution of:

$$\begin{aligned} -\nabla.(a_N(\omega,.)\nabla u_N(\omega,.)) &= f(x) \text{ on } D \\ u_N(\omega,.) &= 0 \text{ on } \partial D. \end{aligned}$$

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- Our aim is to **estimate the error** commited by approximating u by u_N.

Strong convergence of a_N to a

Assumptions:

- the eigenfunctions b_n are continuously differentiable with $\|b_n\|_\infty \leq C$ and $\|\nabla b_n\|_\infty \leq Cn^a$ for some $a \geq 0$,
- $\sum_{n\geq 1} \lambda_n n^b < +\infty$ for some $b > 0$.

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Strong convergence of g_N to g

- Kolmogorov theorem yields:
  $\forall p > 0$, $\forall 0 < \alpha < \min\{b, 2a\}$

$$||g_N - g||_{L^p(\Omega,\mathcal{C}^0(\bar{D}))} \leq A_{\alpha,p}\sqrt{\sum_{n>N} \lambda_n n^\alpha} \quad \forall N \in \mathbb{N}.$$

- Borel-Cantelli lemma yields:
for almost all ω, $g_N \underset{\mathcal{C}^0(\bar{D})}{\longrightarrow} g$ and so $a_N \underset{\mathcal{C}^0(\bar{D})}{\longrightarrow} a$ as $N \to +\infty$.

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  for almost all ω, $g_N \underset{\mathcal{C}^0(\bar{D})}{\longrightarrow} g$ and so $a_N \underset{\mathcal{C}^0(\bar{D})}{\longrightarrow} a$ as $N \to +\infty$.
- We define $a_N^{min}(\omega) = \min\limits_{x \in \bar{D}} a_N(\omega, x)$ and $a_N^{max}(\omega) = \max\limits_{x \in \bar{D}} a_N(\omega, x)$ a.s.
  Then Fernique theorem yields that for all $p > 0$,

$$\left\| \frac{1}{a_N^{min}} \right\|_{L^p(\Omega)} \leq B_p \text{ and } \|a_N^{max}\|_{L^p(\Omega)} \leq B_p \quad \forall N \in \mathbb{N}.$$

Strong convergence of u_N to u

$\forall p > 0$, $\forall 0 < \alpha < \min\{b, 2a\}$

$$\|a_N - a\|_{L^p(\Omega,\mathcal{C}^0(\bar{D}))} \leq C_{\alpha,p} \sqrt{\sum_{n>N} \lambda_n n^\alpha} \quad \forall N \in \mathbb{N}.$$

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Theorem

$\forall p > 0$, $\forall 0 < \alpha < \min\{b, 2a\}$

$$\|u_N - u\|_{L^p(\Omega, H_0^1(D))} \leq F_{\alpha,p} \sqrt{\sum_{n>N} \lambda_n n^\alpha} \quad \forall N \in \mathbb{N}.$$

Weak convergence of u_N to u

Theorem

There exists a constant C such that for any $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ

$$\|\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]\|_{H_0^1} \leq C_\varphi \sum_{n>N} \lambda_n.$$

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Remark: The weak order is twice the strong order.

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Remark: The weak order is twice the strong order.

Sketch of the proof: We recall that: $u_N(\omega, x) = u_N(Y_1(\omega), ..., Y_N(\omega), x)$.
For any N, for any multi-index $\alpha \in \mathbb{N}^N$, and any $y \in \mathbb{R}^N$, we have

$$\left\| \frac{\partial^\alpha u_N(y,x)}{\partial y^\alpha} \right\|_{H_0^1(D)} \leq k_{|\alpha|} \sqrt{\frac{a_N^{max}(y)}{a_N^{min}(y)}} | \, u_N(y)\|_{H_0^1} C^{|\alpha|} \prod_{i \in \mathbb{N}} \sqrt{\lambda_i^{\alpha_i}},$$

where $k_{|\alpha|}$ is a constant independent of N and depending only on the length of α.

Formally, in the particular case of the expected value, we have:

$$u(\omega, x) - u_N(\omega, x)$$
$$= u(Y_1(\omega), ..., Y_N(\omega), Y_{N+1}(\omega), ..., x) - u(Y_1(\omega), ..., Y_N(\omega), 0, ..., x)$$

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& = \sum_{i>N} \frac{\partial u}{\partial y_i}(Y_1(\omega), \ldots, Y_N(\omega), 0, \ldots, x) Y_i(\omega) \\
& + \frac{1}{2} \sum_{i \neq j > N} \frac{\partial^2 u}{\partial y_i \partial y_j}(Y_1(\omega), \ldots, Y_N(\omega), 0, \ldots, x) Y_i(\omega) Y_j(\omega)
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&+ \frac{1}{2} \sum_{i>N} \frac{\partial^2 u}{\partial y_i^2}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x) Y_i(\omega)^2 + ...
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The independence of the Y_i yields:

$$
\mathbb{E}[u - u_N](x) = 0 + \frac{1}{2} \sum_{i>N} \mathbb{E}\left[\frac{\partial^2 u}{\partial y_i^2}(Y_1(\omega), ..., Y_N(\omega), 0, ..., x)\right] + ...
$$

Example: the 1D exponential covariance case

We take $D = (0,1)$ and $cov[g](x,y) = \sigma^2 e^{-\frac{|x-y|}{l}}$ where l is the correlation length. Then we have analytic expressions for the eigenvalues λ_n and the eigenfunctions b_n, in particular :

- $\lambda_n \underset{n \to +\infty}{\sim} \frac{2\sigma^2}{l\pi^2 n^2}$
- $\forall n \in \mathbb{N}$, $\|b_n\|_\infty \leq C$ and $\|b_n'\|_\infty \leq Cn$.

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Proposition (Strong convergence result)

$\forall p > 0$, $\forall 0 < \alpha < 1$

$$\|u_N - u\|_{L^p(\Omega, H_0^1(D))} \leq F_{\alpha,p} N^{\frac{\alpha-1}{2}} \quad \forall N \in \mathbb{N}.$$

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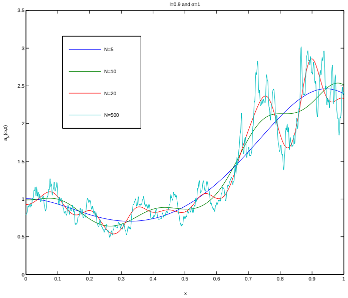
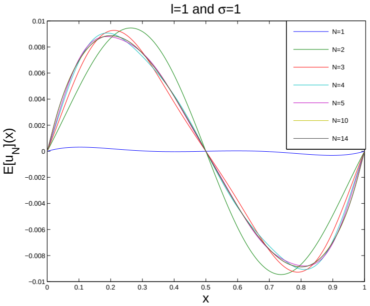
Proposition (Weak convergence result)

For any $\varphi \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ whose derivatives are bounded by a constant C_φ

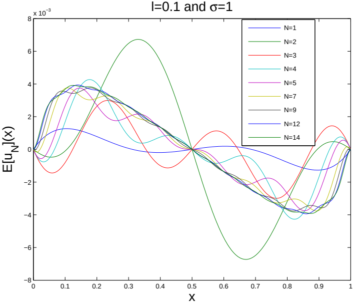
$$\|\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]\|_{H_0^1(D)} \leq \frac{C_\varphi}{N}.$$

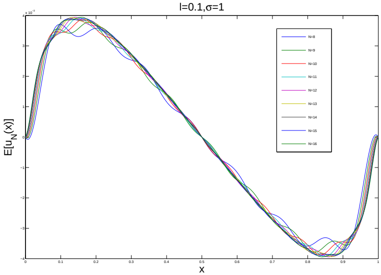




$a_N(\omega, x)$ for different values of N

$\mathbb{E}[u_N(x)]$ for different values of N
here we have $\|E[u - u_N]\|_{L^2(D)} \simeq \frac{c}{N^{2.7}}$.





$\mathbb{E}[u_N(x)]$ for different values of N, in the case where $l = 0.1$, $\sigma = 1$.

Example: the 2D exponential covariance case

We take $D = (0,1)^2$ and $cov[g](x,y) = \sigma^2 e^{-\frac{\|x-y\|_1}{l}}$ where l is the correlation length.

The set of the eigenpairs (λ_n, b_n) is obtained as the tensor product of the 1D eigenpairs.

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Proposition (Strong convergence result)

$\forall p > 0$, $\forall 0 < \alpha < 1$

$$\|u_N - u\|_{L^p(\Omega, H_0^1(D))} \leq F_{\alpha,p} N^{\frac{\alpha-1}{2}} \quad \forall N \in \mathbb{N}.$$

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Proposition (Weak convergence result)

For any $\varphi \in \mathcal{C}^4(\mathbb{R}, \mathbb{R})$ whose derivatives are bounded by a constant C_φ, for any $\varepsilon > 0$, we get

$$\|\mathbb{E}_\omega[\varphi(u) - \varphi(u_N)]\|_{H_0^1(D)} \leq \frac{C_{\varphi,\varepsilon}}{N^{1-\varepsilon}}.$$

Example: the analytic covariance case

We suppose that $cov[g]$ is analytic on D^2, then we have

Theorem (Schwab, Todor)

- $$\lambda_n \leq c_1 e^{-c_2 n^{1/d}} \quad \forall n \in \mathbb{N}$$
- *for any $s > 0$ there exists a constant c_s such that,*

$$\|b_n\|_\infty \leq c_s |\lambda_n|^{-s} \textit{ and } \|b_n'\|_\infty \leq c_s |\lambda_n|^{-s} \quad \forall n \in \mathbb{N}.$$

We have then strong and weak convergence results, analogous to the previous results.

Proposition (Strong convergence result)

For any $0 < s < \frac{1}{2}$, *and* $p > 0$

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq H_{s,p} \sqrt{\sum_{n>N} \lambda_n^{1-2s}} \quad \forall N \in \mathbb{N}$$

therefore

$$\|u - u_N\|_{L^p(\Omega, H_0^1(D))} \leq I_{d,s,p} N^{\frac{d-1}{2d}} e^{-\frac{c_2(1-2s)}{2} N^{1/d}} \quad \forall N \in \mathbb{N}$$

Proposition (Weak convergence result)

For any $0 < s < \frac{1}{2}$*, for all* $\varphi \in \mathcal{C}^4(\mathbb{R},\mathbb{R})$ *whose derivatives are bounded by a constant* C_φ*, we have:*

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq J_s C_\varphi \sum_{n>N} \lambda_n^{1-2s} \quad \forall N \in \mathbb{N}$$

therefore

$$\|\mathbb{E}[\varphi(u_N) - \varphi(u)]\|_{H_0^1(D)} \leq K_{d,s} C_\varphi N^{\frac{d-1}{d}} e^{-c_2(1-2s)N^{1/d}} \quad \forall N \in \mathbb{N}.$$

- Let u_N^h be the approximation of u_N in a standard linear finite element space with a regular triangulation $\mathcal{T}_h$.
- For any $y \in \mathbb{R}^N$, $u_N(y) \in H^2(D)$ with

$$\|u_N(y)\|_{H^2(D)} \leq C \frac{\|f\|_{L^2(D)}}{a_N^{min}(y)} \left(1 + \frac{a_N^{max}(y)}{a_N^{min}(y)}\right) \left(1 + \frac{a_N^{max}(y)}{a_N^{min}(y)} + \frac{\|a_N'(y)\|_{L^\infty(D)}}{a_N^{min}(y)}\right).$$

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- Therefore, thanks to Céa lemma and the usual approximation result, we get

$$\|u_N(y) - u_N^h(y)\|_{H_0^1(D)} \leq$$
$$C\sqrt{\frac{a_N^{max}(y)}{a_N^{min}(y)}} \frac{\|f\|_{L^2(D)}}{a_N^{min}(y)} \left(1 + \frac{a_N^{max}(y)}{a_N^{min}(y)}\right) \left(1 + \frac{a_N^{max}(y)}{a_N^{min}(y)} + \frac{\|a_N'(y)\|_{L^\infty(D)}}{a_N^{min}(y)}\right) h.$$

Proposition

For any $p > 0$, we have

$$\|u_N - u_N^h\|_{L^p(\Omega, H_0^1(D))} \leq C_{N,p} h.$$

If moreover the eigenfunctions $b_n \in \mathcal{C}^2$ and there exists $0 < \theta < 1$ such that

$$\sum_{n \geq 1} \lambda_n \|Db_n\|_\infty^{2(1-\theta)} \|D^2 b_n\|_\infty^{2\theta} < \infty,$$

then the constant $C_{N,p}$ can be chosen independent of N.

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Proposition (J.C,R.Scheichl,A.Teckentrup)

For any $0 < s < 1/2$, $p > 0$, we have

$$\|u_N - u_N^h\|_{L^p(\Omega, H_0^1(D))} \leq C_{s,p} h^s.$$