Quantitative stability results for Vlasov systems

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The N-body problem

- Newton's equations for N interacting bodies

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = \sum_{j \neq i} \nabla V(x_i(t) - x_j(t)).$$

- Newton or Coulomb potential

$$V(r) = \pm \frac{1}{r}.$$

- For N large, a statistical description is more appropriate : Distribution function of bodies : $f(t, x, v)$.

The classical Vlasov-Poisson equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, \qquad f(t=0,x,v) = f_0(x,v)$$

$$\phi_f(t,x) = \frac{\gamma}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_f(t,y)}{|x-y|} dy, \quad \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v) dv.$$

Poisson equation : $\Delta \phi_f = \gamma \rho_f$.

- Collisionless gravitational systems, $\gamma = +1$. : galaxies, star clusters, etc.
- Plasma case , $\gamma = -1$. : charged particles with Coulomb interactions.

Relativistic effects

- Relativistic VP : replace v by $\frac{v}{\sqrt{1+|v|^2}}$: Rein-Hadzic, L-Méhats-Raphaël, Rigault.
- Vlasov-Manev : replace the interaction potential $\frac{1}{|x-y|}$ by $\frac{1}{|x-y|} + \frac{1}{|x-y|^2}$. L-Méhats-Rigault.
- Vlasov-Einstein : Couple Vlasov with relativistic metrics, Einstein equations : Rendall, Andreasson, Rein, ...

Basic properties

- Conservation of the energy : $\mathcal{H}(f) = E_{kin}(f) - \gamma E_{pot}(f)$

$$E_{kin}(f) = \frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f dx dv, \qquad E_{pot}(f) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_x \phi_f|^2 dx$$

- Conservation of the Casimir functionals $\displaystyle\int_{\mathbb{R}^6} G(f) dx dv$.
- Scaling symmetry : f solution $\Longrightarrow$ $\frac{\mu}{\lambda^2} f\left(\frac{t}{\lambda\mu}, \frac{x}{\lambda}, \mu v\right)$ solution too.
- In the case of spherically symmetric solutions $f(t, |x|, |v|, x \cdot v)$, the angular momentum $\int_{\mathbb{R}^6} |x \times v|^2 f dx dv$ is also conserved.

A class of steady states : The spherical models

Spherically symmetric solutions $f := f(|x|, |v|, x \cdot v)$ to

$$v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0.$$

- Isotropic galactic models :

$$f(x,v) = F\left(\frac{|v|^2}{2} + \phi_f(x)\right).$$

- Anisotropic models :

$$f(x,v) = F\left(\frac{|v|^2}{2} + \phi_f(x), |x \times v|^2\right).$$

In fact the Jeans theorem ensures that all spherically symmetric steady states are of this form (Batt-Faltenbacher-Horst 86) :

Stability of steady states

Spherical perturbations

All anisotropic steady states

$$Q(x,v) = F\left(\frac{|v|^2}{2} + \phi_Q(x), |x \times v|^2\right)$$

which are decreasing functions of the microscopic energy are stable under spherical perturbations.

- Proved in ML-Méhats-Raphaël, 2011.
- Optimal : Non spherical perturbations may give instabilities, Binney-Tremaine.

Stability of steady states

General perturbations

All spherically symmetric steady states depending on the energy only

$$Q(x, v) = F\left(\frac{|v|^2}{2} + \phi_Q(x)\right)$$

which are decreasing functions of the microscopic energy are stable under general perturbations. Proved in ML, F. Méhats, P. Raphaël 2012.

A different important context : If periodic domain in space : Homogeneous steady states :

$$f(x, v) = g_0(|v|).$$

Asymptotic stability under Penrose conditions : Landau damping, Mouhot-Villani.

Related works

- Physics literature : Gardner, Antonov, Lynden-Bell (60'), Doremus-Baumann-Feix (1970'), Kandrup-Signet (1980'), Wiechen, Aly, Perez (1990') ..., Binney-Tremaine.
  Linear stability and formal approaches
- Mathematics literature : Two last decades : Wolansky, Guo, Rein, Dolbeault, Lin, Hadzic, Sanchez, Soler, L-Méhats-Raphël ...
  Non linear stability of Minimizers :

$$\text{Minimize } \mathcal{H}(f), \quad \text{with constraints} \quad \|f\|_{L^1} = M_1,\ \|j(f)\|_{L^1} = M_j.$$

Not sufficient.

Statement of the stability result

(i) $Q(x,v) = F\left(\frac{|v|^2}{2} + \phi_Q(x)\right)$ is $\mathcal{C}^0$ and compactly supported.

(ii) F is C^1 on $]-\infty, e_0[$ with $F' < 0$ and, on $[e_0, +\infty[$, $F(e) = 0$.

Theorem (L, Méhats, Raphaël. 2012)

Orbital stability of Q. *For all $\varepsilon > 0$, for all $M > 0$, there exists $\eta > 0$ such that the following holds true. Let $f_0 \in L^1 \cap L^\infty$, with $f_0 \geq 0$ and $|v|^2 f_0 \in L^1$, be such that*

$$\|f_0 - Q\|_{L^1} < \eta, \quad \mathcal{H}(f_0) \leq \mathcal{H}(Q) + \eta \quad \|f_0\|_{L^\infty} < \|Q\|_{L^\infty} + M,$$

then there exists a translation shift $z(t)$ such that the corresponding weak solution $f(t)$ to VP satisfies : $\forall t \geq 0$,

$$\|(1+|v|^2)(f(t,x,v) - Q(x - z(t), v)\|_{L^1(\mathbb{R}^6)} < \varepsilon.$$

Equimeasurability and Schwarz rearrangement

- **Equimeasurability :** consider the set $\mathrm{Eq}(Q)$ of nonnegative functions $f \in L^1 \cap L^\infty$ that are equimeasurable with Q :

$$\int G(f(x,v))dxdv = \int G(Q(x,v))dxdv, \qquad \forall G$$

or

$$\mathrm{meas}\{f(x,v) > \lambda\} = \mathrm{meas}\{Q(x,v) > \lambda\}, \qquad \forall \lambda \geq 0.$$

- **The standard Schwarz symmetrization.** Let $f \in L^1(\mathbb{R}^d)$, then there exists a unique nonincreasing function $f^* \in L^1(\mathbb{R}_+)$ such that $f^*(|x|)$ is equimeasurable with f.
- if f is a solution of the Vlasov system then :

$$f(t)^* = f(0)^*.$$

Two main steps in the original proof

- Reduce the Hamiltonian to a functional of ϕ only :

$$\mathcal{H}(f) - \mathcal{H}(Q) \geq \mathcal{J}(\phi_f) - \mathcal{J}(\phi_Q) - C\|f^* - Q^*\|_{L^1}.$$

and get Local quantitative control of the potential :

$$\inf_{z\in\mathbb{R}^3} \|\nabla\phi_f - \nabla\phi_Q(\cdot - z)\|_{L^2}^2 \leq C\,[\mathcal{H}(f) - \mathcal{H}(Q) + \|f^* - Q^*\|_{L^1}]$$

For all $f \in \mathcal{E}$ such that ϕ_f is in a neighborhood U of ϕ_Q.

- Local compactness of the full distribution function :
Let f_n be any sequence in the energy space such that ϕ_{f_n} is in U. Assume that

$$f_n^* \to Q^* \ \text{in } L^1, \qquad \mathcal{H}(f_n) \to \mathcal{H}(Q).$$

Then there exists a sequence $z_n \in \mathbb{R}^3$ such that

$$\|(1 + |v|^2)(f_n(x, v) - Q(x - z_n, v)\|_{L^1(\mathbb{R}^6)} \to 0.$$

Goal of this presentation

Make fully quantitative the proof this stability result : Mainly the compactness part of the proof may be replaced by a functional inequality.

For all $f \in \mathcal{E}$, there holds :

$$\|f - Q\|_{L^1}^2 \leq \; C_Q \left[\mathcal{H}(f) - \mathcal{H}(Q) + \|f^* - Q^*\|_{L^1} + \frac{1}{2} \|\nabla \phi_f - \nabla \phi_Q\|_{L^2}^2 \right].$$

Rearrangement with respect to the microscopic energy.

Let $\phi(x)$ be a potential field.
Let $f \in L^1 \cap L^\infty(\mathbb{R}^6)$, then we may define its rearrangement with respect to

$$e(x, v) = \frac{|v|^2}{2} + \phi(x).$$

which we denote $f^{*\phi}$. It is

- a nonincreasing function of $\frac{|v|^2}{2} + \phi(x)$;
- such that $f^{*\phi} \in \mathrm{Eq}(f)$.

Caracterisation : Our steady states are fixed points of this transformation

$$Q^{*\phi_Q} = Q$$

Rearrangement with respect to the microscopic energy.

Explicit construction of $f^{*\phi}$

$$f^{*\phi}(x,v) := f^*\left(a_\phi\left(\frac{|v|^2}{2}+\phi(x)\right)\right)\mathbb{1}_{\frac{|v|^2}{2}+\phi(x)<0}$$

where a_ϕ is the Jacobian function defined by

$$\begin{aligned} a_\phi(e) &= \text{meas}\left\{(x,v)\in\mathbb{R}^6 : \frac{|v|^2}{2}+\phi(x)<e\right\} \\ &= \frac{8\pi\sqrt{2}}{3}\int_0^{+\infty}(e-\phi(x))_+^{3/2}\,dx \end{aligned}$$

The key monotonicity property

Lemma. *Let f be a distribution function and ϕ_f its Poisson potential. Then*

$$\mathcal{H}(f) \geq \mathcal{H}(f^{*\phi_f}).$$

Proof.
Denote $\widehat{f} = f^{*\phi_f}$. We have the decomposition

$$\mathcal{H}(f) = \mathcal{H}(\widehat{f}) + \frac{1}{2}\|\nabla\phi_f - \nabla\phi_{\widehat{f}}\|_{L^2}^2 + \int \left(\frac{|v|^2}{2} + \phi_f\right)(f - \widehat{f})dxdv.$$

By construction of $f^{*\phi_f}$, the green term is nonnegative. This is reminiscent from the following property of the standard Schwarz symmetrization :

$$\int_{\mathbb{R}^3} |x| f(x) dx \geq \int_{\mathbb{R}^3} |x| f^*(x) dx.$$

□

Reduction to a problem on the potential

$$\mathcal{H}(f) \geq -C\|f^* - Q^*\| + \mathcal{J}(\phi_f) + \int \left(\frac{|v|^2}{2} + \phi_f\right)(f - f^{*\phi_f})dxdv.$$

$$\mathcal{J}(\phi) = \int \left(\frac{|v|^2}{2} + \phi(x)\right) Q^{*\phi}(x,v)dxdv + \frac{1}{2}\|\nabla\phi\|_{L^2}^2$$

Two points :

- The red term $\mathcal{J}(\phi_f)$ only depends on the potential ϕ_f, and $\mathcal{J}(\phi_Q) = \mathcal{H}(\phi_Q)$. f^* is preserved by the flow.
- The green term is nonnegative and vanishes when $f = Q^{*\phi_f}$.

$$\mathcal{H}(f) - \mathcal{H}(Q) \geq \mathcal{J}(\phi_f) - \mathcal{J}(\phi_Q) - \text{Invariants}.$$

Study of $\mathcal{J}$ and control of ϕ

$$\mathcal{J}(\phi) = \int \left(\frac{|v|^2}{2} + \phi(x) \right) Q^{*\phi}(x,v) dxdv + \frac{1}{2} \|\nabla \phi\|_{L^2}^2$$

$$Q^{*\phi}(x,v) = Q^* \left(a_\phi \left(\frac{|v|^2}{2} + \phi(x) \right) \right)$$

Proposition. *The quantity* $\mathcal{J}(\phi) - \mathcal{J}(\phi_Q)$ *controls the distance of* ϕ *to the manifold of translated Poisson fields* $\mathcal{M} = \{\phi_Q(\cdot + z), \quad z \in \mathbb{R}^3\}$ *: in the vicinity of* $\mathcal{M}$*, we have*

$$\mathcal{J}(\phi) - \mathcal{J}(\phi_Q) \geq C \inf_{z \in \mathbb{R}^3} \|\nabla \phi - \nabla \phi_Q(\cdot - z)\|_{L^2}^2 \quad \textit{with } C > 0.$$

Proof. Based on a Taylor expansion. We differentiate twice the functional $\mathcal{J}$ with respect to ϕ and study the Hessian : it is nonnegative, and coercive on spherical functions.

First variation of $\mathcal{J}$

We prove that

$$D\mathcal{J}(\phi)(h) = -\int_{\mathbb{R}^3} (\nabla\phi_{Q^{*\phi}} - \nabla\phi) \cdot \nabla h \, dx.$$

Consequence :
Since $Q = Q^{*\phi_Q}$ ("fundamental identity of the steady state"), we have $\phi_{Q^{*\phi_Q}} = \phi_Q$, hence $D\mathcal{J}(\phi_Q)(h) \equiv 0$.
This shows that :

ϕ_Q is a critical point of the function $\mathcal{J}$

.

Second variation of $\mathcal{J}$

We prove that

$$D^2\mathcal{J}(\phi_Q)(h,h) = \int |\nabla h|^2 dx - \int (h(x) - \Pi h(e(x,v)))^2 \left|F'(e(x,v))\right| \, dxdv$$

with $e(x,v) = \frac{|v|^2}{2} + \phi_Q(x)$.
Here Π is the projector on functions of $e(x,v)$:

$$\Pi h(e) = \frac{\int (e - \phi_Q(y))_+^{1/2} \, h(y) dy}{\int (e - \phi_Q(y))^{1/2} \, dy}.$$

Crucial : the quadratic form $D^2\mathcal{J}(\phi_Q)(h,h)$ is coercive, up to the degeneracy induced by the translational invariance.

Coercivity of the Hessian (1/2)

Step 1. On radial functions we have a Poincaré inequality :

$$\int |\nabla h|^2 dx \geq \int (h(x) - \Pi h(x,v))^2 \left|F'(e(x,v))\right| dxdv$$

Remark. This is a new Antonov-like inequality.

Proof. Adaptation of Hörmander's approach for sharp weighted Poincaré inequalities

$$\int_{\mathbb{R}^N} \left(f - \frac{\int f d\mu}{\int d\mu}\right)^2 d\mu \leq C \int_{\mathbb{R}^N} |\nabla f|^2 d\mu, \quad d\mu = e^{-V(x)} dx,$$

under the convexity assumption $\nabla^2 V \geq C_0$.

Coercivity of the Hessian (2/2)

Step 2. Decomposing $h = h_0 + h_1$ with h_0 radial and h_1 orthogonal to radial functions, one gets

$$D^2\mathcal{J}(\phi_Q)(h,h) = D^2\mathcal{J}(\phi_Q)(h_0,h_0) + (\mathcal{L}h_1, h_1)$$

with $\mathcal{L} = -\Delta + V_Q$ and $V_Q(x) = \int |F'(e)| dv$. Moreover, by translation invariance,

$$\mathcal{L}(\nabla\phi_Q) = 0.$$

Hence, since ϕ_Q is monotone increasing, a standard argument based on an expansion on spherical harmonics yields that $\mathcal{L}$ restricted on $(\dot{H}^1_{rad})^\perp$ is nonnegative and its kernel is $\mathrm{Span}\,\{\partial_{x_i}\phi_Q,\ 1 \leq i \leq 3\}$.

A general rearrangement of measurable functions

.

Definition

Let θ be a nonnegative and nonzero measurable function of $\mathbb{R}^d$, $d \geq 1$. Consider the associated Jacobian function :

$$\forall e \in \mathbb{R}, \qquad a_\theta(e) = meas\{x \in \mathbb{R}^d, \theta(x) < e\} \in \mathbb{R}_+ \cap \{+\infty\}.$$

Assume that the nondecreasing function a_θ is a convex $\mathcal{C}^1$ diffeomorphism from $[0, e_{max}[$ to $[0, +\infty[$. For all $f \in L^1(\mathbb{R}^d)$, we define its rearrangement $f^{*\theta}$ with respect to θ by

$$f^{*\theta}(x) = f^*(a_\theta(\theta(x))), \quad \forall x \in \mathbb{R}^d,$$

where f^* is the usual Schwarz rearrangement of f. In particular $f^{*\theta}$ is the only decreasing function of $\theta(x)$ which is equimeasurable with f.

A generalized bathtub inequality.

Bathtub inequality (ML, 2012)

Let θ be as above. Then for any nonnegative function $f \in L^1(\mathbb{R}^d)$ such that

$$K(f^*) = \int_0^{\|f\|_{L^\infty}} a'_\theta[a_\theta(2\mu_f(s))]ds < \infty$$

$$\mu_f(s) = meas\{x \in \mathbb{R}^d, f(x) > s\} = \mu_{f^*}(s),$$

we have

$$\|f - f^{*\theta}\|_{L^1}^2 \leq 4K(f^*) \int_{\mathbb{R}^d} \theta(x)(f(x) - f^{*\theta}(x))dx.$$

A particular case : the Schwarz symmetrization

Take $\theta(x) = |x|^m, \quad m \leq d$. In this case $f^{*\theta}(x) = f^*(|B_d||x|^d)$ is the standard Schwarz symmetrization.

Corollary

For all $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $d \geq 1$, and for all $m \leq d$, there holds

$$\int_{\mathbb{R}^d} |x|^m (f(x) - f^*(x))dx \geq K_d \|f\|_{L^\infty}^{-m/d} \|f\|_{L^1}^{-1+m/d} \|f - f^*\|_{L^1}^2,$$

with

$$K_d = 2^{-1+m/d} \frac{m^2}{4d^2} |B_d|.$$

$|B_d|$ is the measure of the unit ball in $\mathbb{R}^d$.

This provides a refinement of the well-known bathtub inequality of rearrangements : $\int |x| f(x)dx \geq \int |x| f^*(x)dx$.

Application to Vlasov equations

$$\mathcal{H}(f)-\mathcal{H}(Q)=\int_{\mathbb{R}^6}\left(\frac{|v|^2}{2}+\phi_Q\right)(f-Q)dxdv-\frac{1}{2}\|\nabla\phi_f-\nabla\phi_Q\|_{L^2}^2.$$

Assume f is equimeasurable with Q to simplify, and let

$$\theta(x,v)=\frac{|v|^2}{2}+\phi_Q(x).$$

$$\mathcal{H}(f)-\mathcal{H}(Q)\geq 4K(Q^*)\|f-Q\|_{L^1}^2-\frac{1}{2}\|\nabla\phi_f-\nabla\phi_Q\|_{L^2}^2.$$

In summary

- Local quantitative control of the potential

$$\inf_{z\in\mathbb{R}^3} \|\nabla\phi_f - \nabla\phi_Q(\cdot - z)\|^2_{L^2} \leq C\left[\mathcal{H}(f) - \mathcal{H}(Q) + \|f^* - Q^*\|_{L^1}\right]$$

- Quantitative control of the full distribution function For all $f \in \mathcal{E}$, there holds :

$$\|f - Q\|^2_{L^1} \leq C_Q \left[\mathcal{H}(f) - \mathcal{H}(Q) + \|f^* - Q^*\|_{L^1} + \frac{1}{2}\|\nabla\phi_f - \nabla\phi_Q\|^2_{L^2}\right]$$

Some perspectives

- Parallel with incompressible fluids is possible (2D incompressible Euler) : Our approach covers the result by Lin (2004), but a complete stability result is not clear yet.
- Extension to other relativistic models : Vlasov-Einstein in the spherically symmetric and asymptotically flat space time.

THANK YOU !