Shape optimization for the observability of PDEs

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Outlines of this talk

1. Introduction and motivation : about the shape optimization of observability constants
2. Optimal observability for wave and Schrödinger equations
   - Solving of the first problem
   - A randomized criterion
   - Solving of the second problem
3. Optimal observability for the heat equation

N-D wave/Schrödinger equations

- $\hookrightarrow$ (M, g) N-D Riemannian manifold
- $\hookrightarrow$ Ω open bounded connected subset of M
- $\hookrightarrow$ $\omega \subset \Omega$ subset of positive measure
- $\hookrightarrow$ Δ_g Laplace Beltrami operator
- $\hookrightarrow$ $T > 0$ fixed

N-D wave equation

$$\left\{ \begin{array}{ll} y_{tt} - \Delta_g y = 0 & (t,x) \in (0,T) \times \Omega \\ y(0,x) = y^0(x),\ \partial_t y(0,x) = y^1(x) & x \in \Omega. \end{array} \right. \tag{1}$$

$\hookrightarrow$ If $\partial\Omega \neq \emptyset$, Dirichlet or Neumann or mixed Dirichlet-Neumann or Robin boundary conditions on $\partial\Omega$

$$\forall (y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega),$$

$$\exists ! y \in \mathcal{C}^0([0,T], H_0^1(\Omega)) \times \mathcal{C}^1([0,T], L^2(\Omega)), \text{ solution of (1)}$$

Observable variable ($\omega \subset \Omega$ of positive measure)

$$z(t,x) = \chi_\omega(x) y_t(t,x) = \left\{ \begin{array}{ll} y_t(t,x) & \text{if } x \in \omega \\ 0 & \text{else.} \end{array} \right.$$

N-D wave/Schrödinger equations

- $\hookrightarrow$ (M, g) N-D Riemannian manifold
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- $\hookrightarrow$ Δ_g Laplace Beltrami operator
- $\hookrightarrow$ $T > 0$ fixed

N-D Schrödinger equation

$$\begin{cases} iy_t - \Delta_g y = 0 & (t,x) \in (0,T) \times \Omega \\ y(0,x) = y^0(x) & x \in \Omega. \end{cases} \tag{2}$$

$\hookrightarrow$ If $\partial\Omega \neq \emptyset$, Dirichlet or Neumann or mixed Dirichlet-Neumann or Robin boundary conditions on $\partial\Omega$

$$\forall y^0 \in H_0^1 \cap H^2(\Omega),$$

$$\exists ! y \in \mathcal{C}^0([0,T], H_0^1 \cap H^2(\Omega)), \text{ solution of } (2)$$

Observable variable ($\omega \subset \Omega$ of positive measure)

$$z(t,x) = \chi_\omega(x) y_t(t,x) = \begin{cases} y_t(t,x) & \text{if } x \in \omega \\ 0 & \text{else.} \end{cases}$$

Observability of the N-D wave equation

$\hookrightarrow$ Without loss of generality, we consider the wave equation with Dirichlet boundary conditions

Observability inequality

The time T being chosen large enough, how to choose $\omega \subset \Omega$ to ensure that $\forall (y^0, y^1) \in H_0^1(\Omega)(\Omega) \times L^2(\Omega)$

$$C_T \|(y^0, y^1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_0^T \int_\Omega z(t,x)^2 dxdt \text{ ?} \tag{3}$$

Observability of the N-D wave equation

↪ Without loss of generality, we consider the wave equation with Dirichlet boundary conditions

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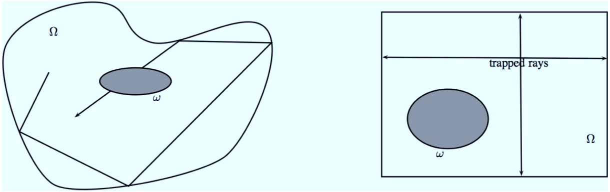
The time T being chosen large enough, how to choose $\omega \subset \Omega$ to ensure that $\forall (y^0, y^1) \in H_0^1(\Omega)(\Omega) \times L^2(\Omega)$

$$C_T \|(y^0, y^1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_0^T \int_\Omega z(t,x)^2 dxdt \ ? \tag{3}$$

- Microlocal Analysis. Bardos, Lebeau and Rauch proved that, roughly in the class of $\mathcal{C}^\infty$ domains, the observability inequality (3) holds iff (ω, T) satisfies the Geometric Control Condition (GCC).



Shape optimization problems

- Observability constant :

$$C_T(\chi_\omega) = \inf_{\substack{y \text{ solution of (1)} \\ (y^0,y^1)\in H_0^1(\Omega)\times L^2(\Omega)}} \frac{\int_0^T \int_\omega y_t(t,x)^2 dxdt}{\|(y^0,y^1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)}}.$$

Shape optimization problems

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Some relevant problems when looking for optimal observability or optimal sensors location

Fix $L \in (0,1)$. We investigate the problem of maximizing

- (Problem 1) either the quantity $G_T(\chi_\omega) = \int_0^T \int_\Omega \chi_\omega(x)|y_t(t,x)|^2 dxdt$, the initial data $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ being fixed,
- (Problem 2) or the observability constant $C_T(\chi_\omega)$

over all possible subset $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

Related problems

Optimal design for control/stabilization problems

1. What is the "best domain" for achieving HUM optimal control ?

$$y_{tt} - \Delta y = \chi_\omega u$$

2. What is the "best domain" domain for stabilization (with localized damping) ?

$$y_{tt} - \Delta y = -k\chi_\omega y_t$$

See works by
- P. Hébrard, A. Henrot : theoretical and numerical results in 1D for optimal stabilization (for all initial data).
- A. Münch, P. Pedregal, F. Periago : numerical investigations of the optimal domain (for one fixed initial data). Study of the relaxed problem.
- S. Cox, P. Freitas, F. Fahroo, K. Ito, ... : variational formulations and numerics.
- M.I. Frecker, C.S. Kubrusly, H. Malebranche, S. Kumar, J.H. Seinfeld, ... : numerical investigations (among a finite number of possible initial data).
- K. Morris, S.L. Padula, O. Sigmund, M. Van de Wal, ... : numerical investigations for actuator placements (predefined set of possible candidates), Riccati approaches.

- ...

Solving of the first problem

Fix $L \in (0,1)$

First Problem

Given $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, we investigate the problem of maximizing

$$G_T(\chi_\omega) = \int_0^T \int_\omega y_t(t,x)^2 dxdt$$

where y is the solution of (1), over all possible subset $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

$\hookrightarrow$ In this maximization problem, the optimal set ω, whenever it exists, depends on the initial data (y^0, y^1).

Solving of the first problem

Spectral rewriting of the first problem

Maximize

$$G_T(\chi_\omega) = \int_\omega \varphi(x)dx \quad \text{where} \quad \varphi(x) = \int_0^T |y_t(t,x)|^2 dt$$

over all possible subsets $\omega \subset \Omega$ of given Lebesgue measure $|\omega| = L|\Omega|$.

Solving of the first problem

Spectral rewriting of the first problem

Maximize

$$G_T(\chi_\omega) = \int_\omega \varphi(x)dx \quad \text{where} \quad \varphi(x) = \int_0^T |y_t(t,x)|^2 dt$$

over all possible subsets $\omega \subset \Omega$ of given Lebesgue measure $|\omega| = L|\Omega|$.

Consequences

- There exists at least one optimal measurable subset $\omega \subset \Omega$;
- Characterization : there exists $\lambda \in \mathbb{R}$ such that, if ω denotes an optimal set, then
  - $\{\chi_\omega = 1\} \subset \{\varphi \geq \lambda\}$;
  - $\{\chi_\omega = 0\} \subset \{\varphi \leq \lambda\}$.

Solving of the first problem

Theorem

Assume that M is an analytic manifold, if $\partial\Omega \neq \emptyset$ is $\mathcal{C}^\infty$, and if y^0 and y^1 have analyticity properties, then the first problem has a unique solution ω that verifies

i ω has a finite number of connected components,

ii ω is semi-analytic,

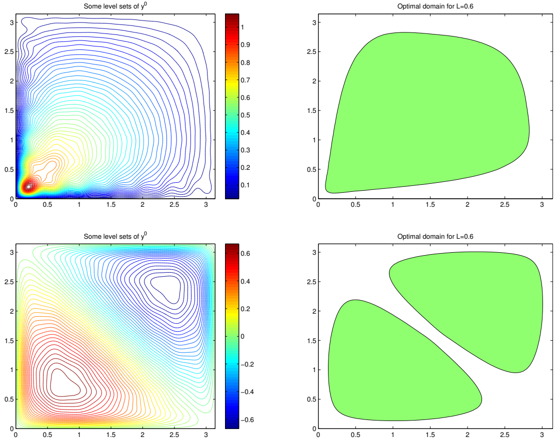
iii ω enjoys the same symmetry properties as Ω.

Remarks

- if y^0 and y^1 have a finite number of nonzero Fourier coefficients (say N), then the optimal set ω has at most $f(N)$ connected components ; S. Mandelbrojt, *Quasi-analycité des séries de Fourier, Ann. Scuola Normale Sup. Pisa, tome 4, no. 3 (1935), 225-229*
- there exist smooth data ($\mathcal{C}^\infty$) for which the set ω has a fractal structure
- initial data for which ω is not unique can be characterized

Solving of the first problem

$$\Omega = [0,\pi]^2,\ L = 0.6,\ T = 3 \text{ and } y^0(x) = \sum_{n,k=1}^{15} a_{n,k} \sin(nx_1)\sin(kx_2),\ y^1 = 0.$$



At the top : $a_{n,k} = \frac{1}{n^2+k^2}$. At the bottom : $a_{n,k} = \frac{1-(-1)^{n+k}}{n^2 k^2}$.

Solving of the second problem

Fix $L \in (0,1)$

Second Problem

We investigate the problem of maximizing the quantity $C_T(\chi_\omega)$ over all possible subsets $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

Spectral expansion of the solution y

$$\forall t \in (0,T),\ \ y(t,\cdot) = \sum_{j=1}^{+\infty} (a_j \cos(\lambda_j t) + b_j \sin(\lambda_j t))\phi_j$$

where

- (λ_j, ϕ_j) denotes the j-th eigenpair of the Laplace-Dirichlet operator on Ω,
- a_j, b_j are determined by the initial conditions.

Solving of the second problem

Fix $L \in (0,1)$

Second Problem

We investigate the problem of maximizing the quantity $C_T(\chi_\omega)$ over all possible subsets $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

Rewriting of $C_T(\chi_\omega)$

$$C_T(\chi_\omega) = \inf_{\substack{(\hat{a}_j),\ (\hat{b}_j)\in\ell^2(\mathbb{C}) \\ \sum_{j=1}^{+\infty}(|\hat{a}_j|^2+|\hat{b}_j|^2)=1}} \int_0^T \int_\omega \left| \sum_{j=1}^{+\infty} (\hat{a}_j e^{i\lambda_j t} - \hat{b}_j e^{-i\lambda_j t})\phi_j(x) \right|^2 dx\, dt$$

- Criterion difficult to handle
- Presence of crossed terms when expanding the square

Solving of the second problem

Fix $L \in (0,1)$

Second Problem

We investigate the problem of maximizing the quantity $C_T(\chi_\omega)$ over all possible subsets $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

- Study of the previous "inf" problem (investigation of the existence, uniqueness of a minimizer).
  $\hookrightarrow$ Linked with the question of the existence of an optimal constant in Ingham's inequality.

Ingham's inequality

Assume that $(\lambda_j) \in \mathbb{R}^n$ verifies $\lambda_{j+1} - \lambda_j \geq \gamma > 0$. Thus, if T is large enough, there exists $C_1,\ C_2 > 0$ s.t. for every $(a_j) \in \ell^2(\mathbb{C})$,

$$C_1 \sum_j |a_j|^2 \leq \int_0^T \left| \sum_j a_j e^{i\lambda_j t} \right|^2 dt \leq C_2 \sum_j |a_j|^2.$$

- Spectral reduction of the criterion ?

Toward a new shape optimization problem

Possible remedies : Randomization of the PDE

$\hookrightarrow$ Random selection of the initial data :

$$y^\nu(t,x) = \sum_{j=1}^{+\infty} \left(\beta_{1,j}^\nu a_j e^{i\lambda_j t} + \beta_{2,j}^\nu b_j e^{-i\lambda_j t} \right) \phi_j(x),$$

where $(\beta_{1,j}^\nu)_{j\in\mathbb{N}^*}$ and $(\beta_{2,j}^\nu)_{j\in\mathbb{N}^*}$ are two sequences of independent Bernoulli random variables on a probability space $(X, \mathcal{A}, \mathbb{P})$, satisfying

$$\mathbb{P}(\beta_{1,j}^\nu = \pm 1) = \mathbb{P}(\beta_{2,j}^\nu = \pm 1) = \frac{1}{2} \qquad \text{and} \qquad \mathbb{E}(\beta_{1,j}^\nu \beta_{2,k}^\nu) = 0$$

for every j and k in $\mathbb{N}^*$ and every event $\nu \in A$.

(see Burq - Tzvetkov, Invent. Math. 2008)

A randomized observability constant

$\hookrightarrow$ We consider the randomized observability inequality

$$C_{T,\text{rand}}(\chi_\omega)\|(y^0,y^1)\|^2_{H^1_0\times L^2} \leq \mathbb{E}\left(\int_0^T \int_\omega y_t^\nu(t,x)^2\, dxdt\right),$$

for all $y^0(\cdot)\in L^2(\Omega)$ and $y^1(\cdot)\in H^{-1}(\Omega)$, where y^ν denotes the solution of the wave equation with random initial data $y^{0,\nu}$ and $y^{1,\nu}$.

A randomized observability constant

$\hookrightarrow$ We consider the randomized observability inequality

$$C_{T,\text{rand}}(\chi_\omega)\|(y^0,y^1)\|^2_{H_0^1\times L^2} \leq \mathbb{E}\left(\int_0^T\int_\omega y_t^\nu(t,x)^2\,dxdt\right),$$

for all $y^0(\cdot)\in L^2(\Omega)$ and $y^1(\cdot)\in H^{-1}(\Omega)$, where y^ν denotes the solution of the wave equation with random initial data $y^{0,\nu}$ and $y^{1,\nu}$.

Proposition

For every measurable set $\omega\subset\Omega$,

$$C_{T,\text{rand}}(\chi_\omega) = T\inf_{j\in\mathbb{N}^*}\int_\omega \phi_j(x)^2\,dx.$$

There holds $C_{T,\text{rand}}(\chi_\omega)\geq C_T(\chi_\omega)$. There are examples where the inequality is strict.

Optimal observability with respect to the domain

Question

What is the “best possible” observation domain ω of given measure ?

Optimal observability with respect to the domain

Question

What is the "best possible" observation domain ω of given measure ?

A new "Second Problem" (energy concentration criterion)

We investigate the problem of maximizing

$$\frac{C_{T,\text{rand}}(\chi_\omega)}{T} = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2\, dx.$$

over all possible subset $\omega \subset \Omega$ of Lebesgue measure $L|\Omega|$.

Solving of the second problem

Spectral rewriting of the second problem

Maximize

$$\frac{C_{T,\mathrm{rand}}(\chi_\omega)}{T} = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2 dx$$

over all possible subsets $\omega \subset \Omega$ of given Lebesgue measure $|\omega| = L|\Omega|$.

Remark. Another justification of the relevance of this criterion.

Proposition

If the spectrum of the Laplace-Dirichlet operator consists of simple eigenvalues, thus

$$\lim_{T\to+\infty} \widetilde{C}_T(\chi_\omega) = \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2 dx.$$

where $\widetilde{C}_T(\chi_\omega)$ stands for the largest constant C in the observability inequality

$$C\|(y^0, y^1)\|^2_{H^1_0\times L^2} \leq \lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_\omega |\partial_t y(t,x)^2| \, dx \, dt.$$

(for all $y^0 \in H^1_0$ and $y^1 \in L^2$)

Solving of the second problem

Relaxation procedure

Second problem

$$\sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} J(\chi_\omega) := \sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} \inf_{j\in\mathbb{N}^*} \int_\omega \phi_j(x)^2 dx$$

- Admissible set for this problem :

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of measure } L|\Omega|\}.$$

- Closure of this set for the weak-star topology of L^∞ :

$$\overline{\mathcal{U}}_L = \left\{ a \in L^\infty(\Omega;(0,1)) \mid \int_\Omega a(x)dx = L|\Omega| \right\}.$$

Relaxed second problem

$$\sup_{a\in\overline{\mathcal{U}}_L} J(a) := \sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2 dx$$

Solving of the second problem

Solving the relaxed second problem

(L^∞-weak Quantum Ergodicity) Assumption

- The sequence $(\phi_j^2)_{j\in\mathbb{N}^*}$ is uniformly bounded in L^∞ norm
- There exists a subsequence such that $\phi_j^2 \rightharpoonup \frac{1}{|\Omega|}$ vaguely as $j \to +\infty$

We have

$$\sup_{a\in\overline{\mathcal{U}}_L} \inf_{j\in\mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2 dx = L \quad \text{(reached with } a = L\text{)}$$

Remarks.

- L^∞-WQE holds true in any flat torus
- if Ω is a convex ergodic billiard with $W^{2,\infty}$ boundary then $\phi_j^2 \rightharpoonup \dfrac{1}{|\Omega|}$ vaguely for a subset of indices of density 1.

Gérard-Leichtnam (Duke Math. 1993), Zelditch-Zworski (CMP 1996), Burq-Zworski (SIAM Rev. 2005), see also Shnirelman, Colin de Verdière,...

Solving the second problem

Gap or no-gap ?

A priori,

$$\sup_{\substack{\omega\subset\Omega \\ |\omega|=L|\Omega|}} J(\chi_\omega) \leq \sup_{a\in\overline{\mathcal{U}}_L} J(a).$$

Remarks in 1D :

- Note that, for every ω, $\int_\omega \sin^2(jx)dx \xrightarrow[j\to+\infty]{} \frac{L\pi}{2}$ as $j\to+\infty$.
- No lower semi-continuity (but upper semi-continuity) of the criterion.
- With $\omega_N = \bigcup_{k=1}^{N} \left[\frac{k\pi}{N+1} - \frac{L\pi}{2N}, \frac{k\pi}{N+1} + \frac{L\pi}{2N}\right]$, one has $\chi_{\omega_N} \rightharpoonup L$ but

$$\lim_{N\to+\infty} J(\omega_N) < L.$$

Solving of the second problem

Theorem 1

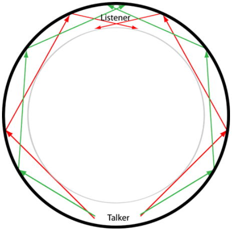
Under L^∞-WQE, there is no gap, that is :

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega \chi_\omega(x)\phi_j(x)^2 \, dx = \sup_{a \in \overline{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega a(x)\phi_j(x)^2 \, dx = L.$$

$\rightarrow$ the maximal value of the time-asymptotic / randomized observability constant is L.

Remark

L^∞-WQE is not a sharp assumption :
the result also holds also true in the Euclidean disk, for which however the eigenfunctions are not uniformly bounded in L^∞ (whispering galleries phenomenon).



Solving of the second problem

(L^p-Quantum Unique Ergodicity) Assumption

- There exists $p > 1$ such that the sequence $(\phi_j^2)_{j\in\mathbb{N}^*}$ is uniformly bounded in L^p norm
- The whole sequence $\phi_j^2 \rightharpoonup \frac{1}{|\Omega|}$ vaguely as $j \to +\infty$.

Introduce the subset of $\mathcal{U}_L$, consisting of characteristic functions of Jordan-measurable subsets ω of Ω, that is

$$\mathcal{U}_L^b = \{\chi_\omega \in \mathcal{U}_L \ \mid \ |\partial\omega| = 0\}$$

Theorem 2

Under L^p-QUE,

$$\sup_{\chi_\omega \in \mathcal{U}_L^b} \inf_{j\in\mathbb{N}^*} \int_\Omega \chi_\omega(x)\phi_j(x)^2 \, dx = L.$$

Remark : The result holds as well if one replaces $\mathcal{U}_L^b$ with either the set of open subsets having a Lipschitz boundary, or with a bounded perimeter.

On the QUE assumption

Quantum Unique Ergodicity property (QUE) in multi-D

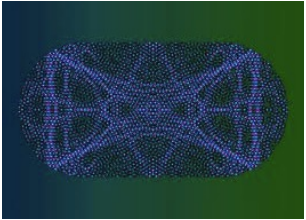
- true in 1D, since $\phi_j(x) = \sqrt{\frac{2}{\pi}}\sin(jx)$ on $\Omega = [0, \pi]$
- Gérard-Leichtnam (Duke Math. 1993), Burq-Zworski (SIAM Rev. 2005) : if Ω is a convex ergodic billiard with $W^{2,\infty}$ boundary then $\phi_j^2 \rightharpoonup \dfrac{1}{|\Omega|}$ vaguely for a subset of indices of density 1.
- Strictly convex billiards sufficiently regular are not ergodic (Lazutkin, 1973).
  Rational polygonal billiards are not ergodic.
  Generic polygonal billiards are ergodic (Kerckhoff-Masur-Smillie, Ann. Math. '86).
- There exist some convex sets Ω (stadium shaped) that satisfy QE but not QUE (Hassell, Ann. Math. 2010)
- QUE conjecture (Rudnick-Sarnak 1994) : every compact manifold having negative sectional curvature satisfies QUE.

On the QUE assumption

Energy concentration phenomena

Hence in general this assumption is related with ergodic / concentration / entropy properties of eigenfunctions.
See Snirelman, Sarnak, Bourgain-Lindenstrauss, Colin de Verdière, Anantharaman, Nonenmacher, ...

If this assumption fails, we may have scars : energy concentration phenomena (there can be exceptional subsequences converging to other invariant measures, like, for instance, measures carried by closed geodesics : scars)



A truncated problem

Assume that there is no gap, i.e.

$$\sup_{\substack{\omega\subset\Omega\\|\omega|=L|\Omega|}} J(\chi_\omega) = \sup_{a\in\overline{\mathcal{U}}_L} J(a) =: J.$$

$$\Rightarrow \lim_{N\to+\infty} \sup_{\chi_\omega\in\mathcal{U}_L} \inf_{1\leq j\leq N} \int_\Omega \chi_\omega(x)\phi_j(x)^2\, dx = J.$$

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Theorem (YP-Sigalotti - COCV 2009)

Let $L\in(0,1)$. The shape optimization problem

$$\sup_{\chi_\omega\in\mathcal{U}_L} \inf_{1\leq j\leq N} \int_\Omega \chi_\omega(x)\phi_j(x)^2\,dx$$

has a unique solution ω_N^*.

$\hookrightarrow$ Convergence of $(\chi_{\omega_N^*})_{N\in\mathbb{N}^*}$ to a minimizer of the second problem.

Solving the truncated second problem

The 1-D case - $\Omega = (0, \pi)$

Truncated second problem

$$\sup_{\substack{\omega\subset[0,\pi] \\ |\omega|=L\pi}} \inf_{1\leq j\leq N} \int_\omega \sin^2(jx)dx$$

Theorem (Hébrard-Henrot and YP-Trélat-Zuazua)

This problem has a unique solution ω^N, satisfying

- ω^N is the union of at most N intervals
- ω^N is symmetric w.r.t. $\pi/2$
- there exists η_N such that $\omega^N \subset [\eta_N, \pi - \eta_N]$
- there exists $L_N \in (0, 1]$ such that, for every $L \in (0, L_N]$,

$$\int_{\omega_N} \sin^2 x dx = \int_{\omega_N} \sin^2(2x)dx = \cdots = \int_{\omega_N} \sin^2(Nx)dx$$

Solving the truncated second problem

The 1-D case - $\Omega = (0, \pi)$

Truncated second problem

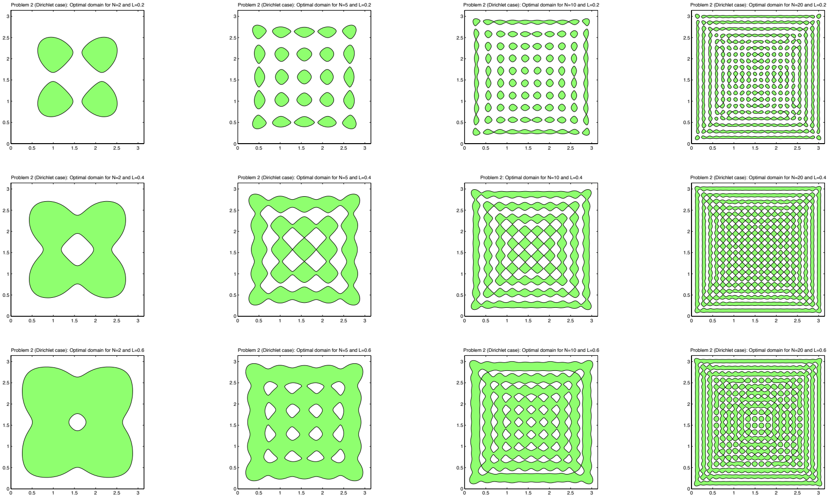
$$\sup_{\substack{\omega\subset[0,\pi]\\ |\omega=L\pi|}} \inf_{1\leq j\leq N} \int_{\omega} \sin^2(jx)dx$$

- Equality of the criteria $\longrightarrow$ the optimal domain ω^N concentrates around the points $\frac{k\pi}{N+1}$, $k = 1, \cdots, N$
- Spillover phenomenon : the best domain ω^N for the first N modes is the worst possible for $N+1$ modes.

The proof appears unexpectedly difficult...

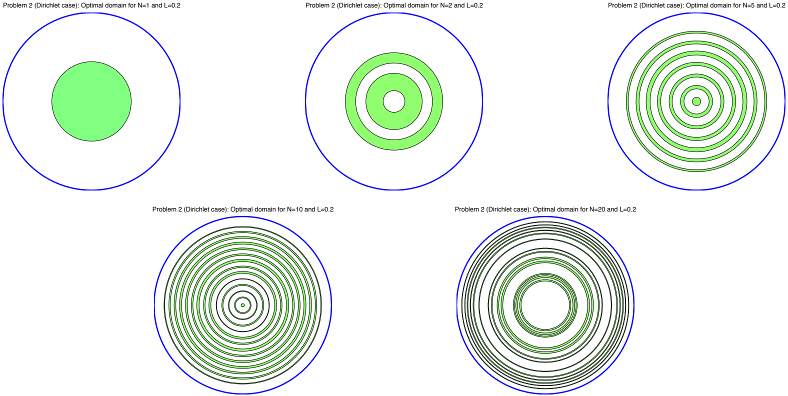
Several numerical simulations : $\Omega = [0, \pi]^2$

For 4, 25, 100 and 500 eigenmodes and $L \in \{0.2, 0.4, 0.6\}$



Several numerical simulations : $\Omega =$ unit disk

$L = 0.2$, for 1, 4, 25, 100 and 400 eigenmodes



N-D heat equation

$$\begin{cases} y_t - \Delta_g y = 0 & (t,x) \in (0,T) \times \Omega \\ y(t,x) = 0 & t \in [0,T],\ x \in \partial\Omega \\ y(0,x) = y^0(x) & x \in \Omega. \end{cases}$$

$\hookrightarrow\ \exists! y \in \mathcal{C}^0(0,T; H^2 \cap H^1_0(\Omega)) \cap \mathcal{C}^0(0,T; L^2(\Omega))$

Observable variable ($\omega \subset \Omega$ of positive measure)

$$z(t,x) = \chi_\omega(x) y(t,x)$$

Observability inequality

$$C_T(\chi_\omega)\|y(T,\cdot)\|^2_{L^2(\Omega)} \leq \int_0^T \int_\omega y(t,x)^2\, dxdt,$$

N-D heat equation

Randomization procedure

$\hookrightarrow$ Randomization of the observability constant :

$$C_{T,\text{rand}} \|y_\nu(T,\cdot)\|^2_{L^2(\Omega)} \leq \mathbb{E}\left(\int_0^T \int_\omega y_\nu(t,x)^2 \, dx \, dt\right),$$

for all $y(T,\cdot) \in L^2(\Omega)$, where y_ν denotes the solution of the wave equation with the random intial data y_ν^0

Proposition

$$C_{T,\text{rand}}(\chi_\omega) = \inf_{j\in\mathbb{N}^*} \gamma_j \int_\Omega \chi_\omega(x)\phi_j(x)^2 \, dx,$$

where $\gamma_j = \dfrac{e^{2\lambda_j T} - 1}{2\lambda_j}$.

N-D heat equation

An existence result

Theorem

Assume that

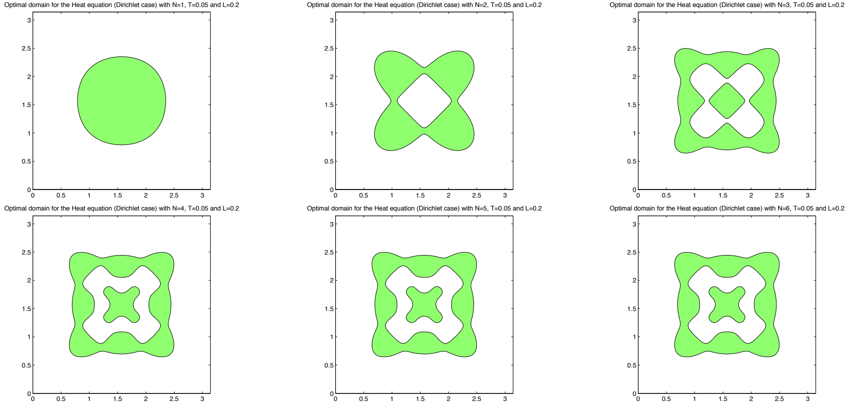
- either Ω satisfies the L^p-(QUE) property,
- or Ω satisfies the L^∞-(WQE) property,
- or Ω is a flat torus.

There exists $N_0 \in \mathbb{N}^*$ such that

$$\max_{a\in\overline{\mathcal{U}}_L} \min_{1\leq j} \gamma_j \int_\Omega a(x)\phi_j(x)^2 dx = \max_{\chi_\omega \in \mathcal{U}_L} \min_{1\leq j\leq N_0} \gamma_j \int_\omega \phi_j(x)^2 dx.$$

$\hookrightarrow$ Stabilization of the optimal domain in the truncation procedure...

Several numerical simulations : $\Omega = [0,\pi]^2$, $T = 0.05$ and $L = 0.2$ for $N \in \{1,2,3,4,5,6\}$



Stabilization from $N = 4$ (i.e. 16 eigenmodes)

Conclusion of this talk

- Ongoing work (with P. Jounieaux and E. Trélat) : optimal design for boundary observability or control
  Ω being assumed bounded and its boundary $\mathcal{C}^2$, maximize

$$\inf_{j\in\mathbb{N}^*} \frac{1}{\lambda_j(\Omega)} \int_\Sigma \left|\frac{\partial \phi_j}{\partial n}\right|^2 dx$$

  over all possible subsets $\Sigma \subset \partial\Omega$ of given Hausdorff measure.
- Discretization issues (with E. Trélat and E. Zuazua) : do the numerical designs converge to the continuous optimal design as the mesh size tends to 0 ?

Y. Privat, E. Trélat, E. Zuazua, *Optimal observation of the one-dimensional wave equation*, to appear in J. Fourier Analysis Appl.

Y. Privat, E. Trélat, E. Zuazua, *Optimal location of controllers for the one-dimensional wave equation*, to appear in Ann. Inst. H. Poincaré.

Y. Privat, E. Trélat, E. Zuazua, *Optimal observability of wave and Schrödinger equations in ergodic domains*, Preprint (2012).

Kenavo ha trugarez

Au revoir et merci