Asymptotic diagonalization of Hilbert-Schmidt operators

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Joint work with N. Raymond

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A little history

Hint around the power method : Consider $A \in \mathsf{M}_n(\mathbb{C})$, $x_0, y_0 \in \mathbb{C}^n$.
For z sufficiently large :

$$f(z) = \langle x_0, (zI - A)^{-1} y_0 \rangle = \sum_{\nu=0}^{+\infty} \frac{\overbrace{\langle x_0, A^\nu y_0 \rangle}^{=: s_\nu}}{z^{\nu+1}} = \frac{\langle x_0, \mathsf{adj}(zI - A) y_0 \rangle}{\mathsf{det}(zI - A)}.$$

Under some assumptions on x_0, y_0 and on A:

$$\frac{s_{\nu+1}}{s_\nu} \underset{\nu\to\infty}{\longrightarrow} \lambda_1, \text{ eigenvalue of maximum modulus.}$$

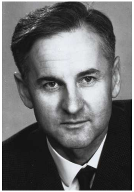
More generaly : *Being given the sequence of moments (s_ν), find all the eigenvalues of a matrix A ?*

Hadamard

1892 Hadamard extended the above idea to obtain any of the pole of a meromorphic function from its moments, using the sequence of Hankel determinants of the form $|s_{\nu+i+j}|_{0 \le i,j \le k}$.

1931 Aitken rediscovered a more algorithmic version of the method.

1954 Rutishauser introduced **qd (quotient-difference) algorithm**, avoiding Hankel determinants, reducing ill-posedness of the method, and then reformulated (first for tridiagonal matrices) into the **LR algorithm** (Gauss LU decomposition)



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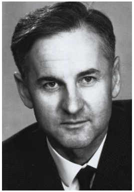


Rutishauser

1961 Francis, Kublanovskaya : **QR algorithm**

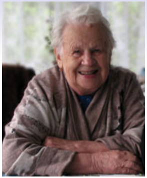
$$Q_k R_k := A_k, \quad A_{k+1} := R_k Q_k$$

$$A_{k+1} = Q_k^\star A_k Q_k = (Q_1 \dots Q_k)^\star A (Q_1 \dots Q_k)$$

$$\text{spec } A_k = \text{spec } A$$

Francis, Wilkinson : Fast variants with shifts

$$Q_k R_k := A_k - \sigma_k I_n, \quad A_{k+1} := R_k Q_k + \sigma_k I_n$$



Kublanovskaya

Francis

1968 Colette Lebaud :

Remarques sur la convergence de la méthode Q.R., Publications mathématiques et informatique de Rennes, Tome (1967-1968), Exposé no. 5, p.1–17.

PhD Univ. Rennes : *Contribution à l'étude de l'algorithme QR*, 1971.

Lebaud

Common convergence results for the QR algorithm

Theorem

Let $A \in \mathsf{M}_n(\mathbb{C})$ *and* $P \in \mathsf{GL}_n(\mathbb{C})$ *such that* $PAP^{-1} = diag(\lambda_1, \ldots, \lambda_n) = D$ *with*

$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n| > 0.$$

Assume moreover that P *admits a LU factorization. Then*

$$A_k \underset{k\to\infty}{\longrightarrow} \begin{pmatrix} \lambda_1 & & (\star)_k \\ & \ddots & \\ (0) & & \lambda_n \end{pmatrix}.$$

The lower part converge as $\mathcal{O}(\mu^k)$ *with* $\mu = \max_{1\le\ell\le n-1} \left|\frac{\lambda_{\ell+1}}{\lambda_\ell}\right|$.

- In presence of "same modulus" distinct eigenvalues: convergence "by block".
- Iterations preserve symmetry and Hessenberg structure.
- Convergence is quadratic for the Wilkinson shifted version, and then cubic for symmetric matrices.
- The algorithm is equivalent to the orthogonal simultaneous power iteration (this makes the assumption $P = LU$ and improved convergence rates more intelligible)

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Continuous dynamical systems

1988 Chu and Norris, *Isospectral flows and abstract matrix factorizations*, SIAM J. Numer. Anal.

1991 Brockett, *Dynamical systems that sort lists, diagonalize matrices and solve linear programming problems*, Linear Algebra Appl.

1994 Wegner, *Flow equations for Hamiltonians*, Ann. Physik.

2010 Bach and Bru, *Rigorous foundations of the Brockett-Wegner flow for operators*, J. Evol. Equ.

2016 Bach and Bru, *Diagonalizing quadratic bosonic operators by non-autonomous flow equation*, Memoirs AMS.

References for some history :

Gutknecht and Parlett, *From qd to LR, or, How were the qd and LR algorithms discovered?*, IMA J. Numer. Anal. 31 (2011).

Golub and van der Vorst, *Eigenvalue computation in the 20th century*, J. Comput. Appl. Math. 123 (2000).

Our aim :

Get a unified proof for different bracket flow ODEs : $H' = [H, G(H)]$,
acting on Hilbert-Schmidt operators,
and understand the QR method !

Outline

Introduction

Main result

Bracket flows
- Global existence and isospectrality
- Quadratic bracket flows
- Several examples

The Toda's choice
- A more specific result
- Relation with the discrete QR algorithm
- Numerical examples

Some notations

$\mathcal{L}(\mathcal{H})$ bounded operators on a separable Hilbert space $\mathcal{H}$

$\mathcal{S}(\mathcal{H}), \mathcal{A}(\mathcal{H}), \mathcal{U}(\mathcal{H})$ symmetric, skew-symmetric and unitary operators respectively

$\mathcal{L}_2(\mathcal{H})$ Hilbert-Schmidt (compact) operators on $\mathcal{H}$, with the norm $\|\cdot\|_{\mathsf{HS}}$:

$$\|H\|_{\mathsf{HS}}^2 = \mathrm{tr}(H^\star H) = \sum_{n\geq 0} \|He_n\|^2 = \sum_{n=0}^{+\infty} |\lambda_n(H)|^2$$

$\mathcal{S}_2(\mathcal{H}) = \mathcal{S}(\mathcal{H}) \cap \mathcal{L}_2(\mathcal{H})$

$\mathcal{A}_2(\mathcal{H}) = \mathcal{A}(\mathcal{H}) \cap \mathcal{L}_2(\mathcal{H})$

$(e_n)_{n\in\mathbb{N}}$ a Hilbert basis of $\mathcal{H}$

$h_{i,j} = \langle He_i, e_j\rangle$, for any $H \in \mathcal{L}(\mathcal{H})$

$\mathcal{D}(\mathcal{H})$ bounded diagonal operators with respect to the Hilbert basis:

$$T \in \mathcal{D}(\mathcal{H}) \Leftrightarrow \forall n \in \mathbb{N},\ \exists \lambda_n \in \mathbb{R},\ Te_n = \lambda_n e_n$$

$E_{i,j}$ $(i \leq j)$ canonical Hilbert basis of $\mathcal{S}_2(\mathcal{H})$

$E_{i,j}^{\pm}$ $(i < j)$ canonical Hilbert basis of $\mathcal{A}_2(\mathcal{H})$

Main result

Consider $H \in \mathcal{C}^1(\mathbb{R}, \mathcal{L}_2(\mathcal{H}))$ such that

i) the function $t \mapsto \|H(t)\|_{\mathrm{HS}}$ is bounded,

ii) the family $(h_{i,j})_{i,j\in\mathbb{N}}$ is balanced (symmetry-like property, uniform in t)

iii) there exists a pointwise bounded and balanced family of measurable functions $(g_{i,j})_{i,j\in\mathbb{N}}$ defined on $\mathbb{R}$ such that

$$\forall t \in \mathbb{R},\ \forall i \in \mathbb{N}, \quad h'_{i,i}(t) = \sum_{j=0}^{+\infty} g_{i,j}(t) |h_{i,j}(t)|^2 .$$

Suppose that there exists $T > 0$ and a sign sequence $(\epsilon_\ell)_{\ell\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$, such that

$$\forall t \geq T,\ \forall k, \ell \in \mathbb{N},\ k \geq \ell,\ \epsilon_\ell g_{\ell,k}(t) \geq 0 . \qquad \text{(sign)}$$

Then we have the integrability property

$$\forall \ell \in \mathbb{N}, \quad \sum_{j=0}^{+\infty} \int_T^{+\infty} |g_{\ell,j}(t)||h_{\ell,j}(t)|^2 \mathrm{d}t < +\infty ,$$

and the diagonal terms $h_{\ell,\ell}(t)$ converges at infinity.

Ideas of the proof

- Proof by induction on $\ell \in \mathbb{N}$, using monotonicity arguments
- Limit of the $h_{\ell,\ell}$ coefficient and integrability of the series is controled from the next lemma

Lemma

Consider $h \in \mathcal{C}^1(\mathbb{R}_+)$ a real-valued bounded function, $F \in L^1(\mathbb{R}_+)$ and G a nonnegative measurable function defined on $\mathbb{R}_+$ such that

$$\forall t \geq 0,\ h'(t) = F(t) + G(t).$$

Then, the function G is integrable and h converges to a finite limit at infinity.

$$\int_0^x G(t)\,dt = h(x) - h(0) - \int_0^x F(t)\mathrm{d}t \leq 2\sup_{\mathbb{R}_+}|h| + \int_0^{+\infty} |F(t)|\mathrm{d}t\,.$$

Therefore $G \in L^1(\mathbb{R}_+)$ and $\lim_{x\to+\infty} h(x) = h(0) + \int_0^{+\infty} (F(t)+G(t))\mathrm{d}t\,.$

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Main result (2)

Under the additional semi-uniform lower bound condition

$$\forall \ell \in \mathbb{N},\ \exists c_\ell > 0,\ \forall j \neq \ell,\ \forall t \geq T,\ |g_{\ell,j}(t)| \geq c_\ell\,, \qquad \text{(lower bound)}$$

we have the stronger integrability result

$$\sum_{j\neq\ell} \int_T^{+\infty} |h_{\ell,j}(t)|^2\, dt = \int_T^{+\infty} \|H(t)e_\ell - h_{\ell,\ell}(t)e_\ell\|^2 \mathrm{d}t < +\infty\,.$$

Suppose moreover that, for any $\ell \in \mathbb{N}$, the function $t \mapsto \|H'(t)e_\ell\|$ is bounded, then for any $\ell \in \mathbb{N}$, $H(t)e_\ell$ converges to $h_{\ell,\ell}(\infty)e_\ell$.

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Global existence and isospectrality

Theorem

Suppose $G : \mathcal{S}(\mathcal{H}) \to \mathcal{A}(\mathcal{H})$ *is locally Lipschitz, then any solution to the following Cauchy problem*

$$H' = [H, G(H)]\,, \qquad H(0) = H_0 \in \mathcal{S}_2(\mathcal{H})\,.$$

admits a unique global solution $H \in \mathcal{C}^1(\mathbb{R}, \mathcal{S}_2(\mathcal{H}))$ *and there exists* $U \in \mathcal{C}^1(\mathbb{R}, \mathcal{U}(\mathcal{H}))$ *such that*

$$H(t) = U(t)^\star H_0 U(t), \quad t \in \mathbb{R}.$$

Proof: As long $H(t)$ is defined, consider the following Cauchy problem:

$$\begin{vmatrix} U'(t) = U(t)G(H(t)), \\ U(0) = \mathrm{Id}\,. \end{vmatrix}$$

Then $U(t) \in \mathcal{U}(\mathcal{H})$ and $H(t) = U^\star(t) H_0 U(t)$.

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Then $U(t) \in \mathcal{U}(\mathcal{H})$ and $H(t) = U^\star(t) H_0 U(t)$.

$$\begin{aligned} \tfrac{d}{dt}(UU^\star) &= U'U^\star + UU'^\star \\ &= UG(H)U^\star + U(U\underbrace{G(H)}_{\in \mathcal{A}(\mathcal{H})})^\star = UG(H)U^\star - UG(H)U^\star = 0. \end{aligned}$$

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Then $U(t) \in \mathcal{U}(\mathcal{H})$ and $H(t) = U^\star(t) H_0 U(t)$.

$$\begin{aligned} \tfrac{d}{dt}(UHU^\star) &= U'HU^\star + UH'U^\star + UHU'^\star \\ &= UG(H)HU^\star + U[H, G(H)]U^\star - UHG(H)U^\star \\ &= U\Big(G(H)H - HG(H) + [H, G(H)]\Big)U^\star = 0. \end{aligned}$$

Quadratic bracket flows

Consider $H' = [H, G(H)]$, in the case $G \in \mathcal{L}(\mathcal{S}_2(\mathcal{H}), \mathcal{A}_2(\mathcal{H}))$.

The map G is said to be diagonalizable when there exists a Hilbertian basis $(e_n)_{n\in\mathbb{N}}$ and a skew-symmetric family $(g_{i,j})$ such that $G(E_{i,j}) = g_{i,j}E^{\pm}_{i,j}$.

We then have

$$h'_{i,i}(t) = -2\sum_j g_{i,j}|h_{i,j}(t)|^2,$$

and prove from the main result :

Corollary

Let $G \in \mathcal{L}(\mathcal{S}_2(\mathcal{H}), \mathcal{A}_2(\mathcal{H}))$ be a diagonalizable map such that its matrix-eigenvalue satisfies (sign) and (lower bound).
Then the unique global solution $H \in \mathcal{C}^1(\mathbb{R}, \mathcal{S}_2(\mathcal{H}))$ converges weakly in $\mathcal{S}_2(\mathcal{H})$ to some $H_\infty \in \mathcal{D}_2(\mathcal{H})$.
Moreover, any diagonal term α of H_∞ with multiplicity m is an eigenvalue of H_0 of multiplicity at least m.

Example 1: Brockett's choice

$G_{\mathsf{Br}}(H) = [H, A]$, where $A = \mathrm{diag}(a_1, \ldots) \in \mathcal{D}(\mathcal{H})$, $a_1 > a_2 > \ldots > 0$

- skew-symmetric matrix-eigenvalue :

$$g_{i,j} = a_j - a_i$$

- (sign) assumption :

$$\forall k, \ell \in \mathbb{N},\ k \geq \ell,\ -g_{\ell,k} \geq 0$$

- (lower bound) assumption :

$$\forall \ell \in \mathbb{N},\ \forall j \neq \ell,\ |g_{\ell,j}| \geq \min(|a_\ell - a_{\ell-1}|, |a_\ell - a_{\ell+1}|)$$

Example 2: Toda's choice

$$G_{\mathsf{Tod}}(H) = H^- - (H^-)^\star, \qquad H^- = \sum_{1 \le i \le j} h_{i,j} e_i^* e_j$$

- skew-symmetric matrix-eigenvalue :

$$g_{i,j} = -1, \quad i < j$$

- (sign) assumption :

$$\forall k, \ell \in \mathbb{N},\ k \ge \ell,\ -g_{\ell,k} \ge 0$$

- (lower bound) assumption :

$$\forall \ell \in \mathbb{N},\ \forall j \neq \ell,\ |g_{\ell,j}| \ge 1$$

Example 3: Wegner's choice

$$G_{\text{Weg}}(H) = [H, \text{diag}(H)]$$

- $G_{\text{Weg}} \notin \mathcal{L}\left(\mathcal{S}_2(\mathcal{H}), \mathcal{A}_2(\mathcal{H})\right)$
- but $G_{\text{Weg}}(\mathcal{S}_2(\mathcal{H})) \subset \mathcal{A}_2(\mathcal{H})$ therefore the flow is unitarily equivalent and global.
- diagonal terms follow the dynamic:

$$h'_{i,i} = \sum_{j=0}^{+\infty} \underbrace{(h_{i,i} - h_{j,j})}_{g_{i,j}(t)} |h_{ij}|^2 .$$

- What about (sign) and (lower bound) assumptions ?
  If for large time, the diagonal terms $(h_{\ell,\ell})$ become "sufficiently" distinct, both assumptions are valid.

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More about the Toda flow : $G(H) = H^- - (H^-)^\star$.

Proposition

Assume that $H_0 \in \mathcal{L}_2(\mathcal{H})$ (not necessarily symmetric) is diagonalizable with eigenvalues $(\lambda_j)_{j\geq 0}$ such that

$$\mathrm{Re}\lambda_0 > \mathrm{Re}\lambda_1 > \dots$$

Let us also assume that we may find $P \in \mathcal{L}(\mathcal{H})$ such that $PH_0P^{-1} = \mathrm{diag}(\lambda_j)$ and such that all the minors of P : $P_J = (\langle Pe_i, e_j \rangle)_{0\leq i,j\leq J}$ are invertible.

For $\ell \in \mathbb{N}$, we denote $\delta_\ell = \min_{0\leq j\leq \ell} \mathrm{Re}(\lambda_j - \lambda_{j+1})$, then

$$H(t)e_\ell - \sum_{j=\ell+1}^{+\infty} \langle H(t)e_\ell, e_j \rangle e_j = \lambda_\ell e_\ell + \mathcal{O}\left(e^{-t\delta_\ell}\right).$$

- Find exponentially fast eigenvalues of any Hilbert-Schmidt operator
- If $H(0)$ is symmetric, then $H(t)$ converges to a diagonal operator H_∞ that has exactly the same eigenvalues as H_0
- Even in infinite dimension no eigenvalue is lost in the limit !

Relation with the discrete QR algorithm

1988 Chu and Norris proposed an abstract decomposition of the flow

$$H'(t) = [H(t), G_{\mathsf{Tod}}(H(t))]\,, \qquad H(0) = H_0 \in \mathcal{L}_2(\mathcal{H})\,,$$

$$g_1'(t) = g_1(t) G_{\mathsf{Tod}}(H(t))\,, \qquad g_1(0) = \mathrm{Id}\,,$$

$$g_2'(t) = (H(t) - G_{\mathsf{Tod}}(H(t))) g_2(t)\,, \qquad g_2(0) = \mathrm{Id}\,.$$

so that

$$H(t) = g_1(t)^{-1} H_0 g_1(t) = g_2(t) H_0 g_2^{-1}(t)$$

$$e^{tH_0} = g_1(t) g_2(t)\,, \qquad e^{tH(t)} = g_2(t) g_1(t)\,.$$

- $G_{\mathsf{Tod}}(H) = H_- - H_-^\star \in \mathcal{A}(\mathcal{H})$ therefore $g_1(t) \in \mathcal{U}(\mathcal{H})$
- $H - G_{\mathsf{Tod}}(H) = H_+ + H_-^\star$ is upper triangular and so is $g_2(t)$
- at $t = 1$: $e^{H_0} = g_1(1) g_2(1)$ is a QR factorization of e^{H_0}
  and $e^{H(1)} = g_2(1) g_1(1)$ is then the first iterate of the QR algorithm
- Toda flow $\simeq$ sampling of the discrete algorithm (up to a change in $g_2(0)$ to ensure corresponding normalization choices in the diagonal of $g_2(1)$)
- spec$(\mathrm{e}^{H_0}) = \{\mathrm{e}^{\lambda_j}\}$ with descreasing moduli (convergence rate)
- *Remark* : this can be done for many matrix factorization : QR, LU, Cholesky.

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$$H(t) = g_1(t)^{-1} H_0 g_1(t) = g_2(t) H_0 g_2^{-1}(t)$$

$$e^{tH_0} = g_1(t) g_2(t) \,, \qquad e^{tH(t)} = g_2(t) g_1(t) \,.$$

- $G_{\mathsf{Tod}}(H) = H_- - H_-^\star \in \mathcal{A}(\mathcal{H})$ therefore $g_1(t) \in \mathcal{U}(\mathcal{H})$
- $H - G_{\mathsf{Tod}}(H) = H_+ + H_-^\star$ is upper triangular and so is $g_2(t)$
- at $t = 1$: $e^{H_0} = g_1(1) g_2(1)$ is a QR factorization of e^{H_0}
  and $e^{H(1)} = g_2(1) g_1(1)$ is then the first iterate of the QR algorithm
- Toda flow $\simeq$ sampling of the discrete algorithm (up to a change in $g_2(0)$ to ensure corresponding normalization choices in the diagonal of $g_2(1)$)
- $\mathrm{spec}(e^{H_0}) = \{e^{\lambda_j}\}$ with descreasing moduli (convergence rate)
- *Remark* : this can be done for many matrix factorization : QR, LU, Cholesky.

Relation with the discrete QR algorithm

1988 Chu and Norris proposed an abstract decomposition of the flow

$$H'(t) = [H(t), G_{\mathsf{Tod}}(H(t))]\,, \qquad H(0) = H_0 \in \mathcal{L}_2(\mathcal{H})\,,$$

$$g_1'(t) = g_1(t) G_{\mathsf{Tod}}(H(t))\,, \qquad g_1(0) = \mathrm{Id}\,,$$
$$g_2'(t) = (H(t) - G_{\mathsf{Tod}}(H(t))) g_2(t)\,, \qquad g_2(0) = \mathrm{Id}\,.$$

so that

$$H(t) = g_1(t)^{-1} H_0 g_1(t) = g_2(t) H_0 g_2^{-1}(t)$$

$$e^{tH_0} = g_1(t) g_2(t)\,, \qquad e^{tH(t)} = g_2(t) g_1(t)\,.$$

- $G_{\mathsf{Tod}}(H) = H_- - H_-^\star \in \mathcal{A}(\mathcal{H})$ therefore $g_1(t) \in \mathcal{U}(\mathcal{H})$
- $H - G_{\mathsf{Tod}}(H) = H_+ + H_-^\star$ is upper triangular and so is $g_2(t)$
- at $t = 1$: $e^{H_0} = g_1(1) g_2(1)$ is a QR factorization of e^{H_0}
  and $e^{H(1)} = g_2(1) g_1(1)$ is then the first iterate of the QR algorithm
- Toda flow $\simeq$ sampling of the discrete algorithm (up to a change in $g_2(0)$ to ensure corresponding normalization choices in the diagonal of $g_2(1)$)
- spec$(\mathrm{e}^{H_0}) = \{\mathrm{e}^{\lambda_j}\}$ with descreasing moduli (convergence rate)
- *Remark :* this can be done for many matrix factorization : QR, LU, Cholesky.

Finite dimension case :

- strong convergence to the limit
- alternative : compactness of $\mathcal{U}(\mathcal{H})$
- linearization of the flow in $\mathcal{U}(\mathcal{H})$ around U_∞, parameterized by its Lie algebra $\mathcal{A}(\mathcal{H})$
- $\Rightarrow$ exponential convergence, whatever is the stable manifold in consideration.

Datas for the computations :

- Initial data : $H_0 = Q \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 25 \end{pmatrix} Q^\star$ where $Q = \exp \begin{pmatrix} 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$

- G_{Br} is defined from $A = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

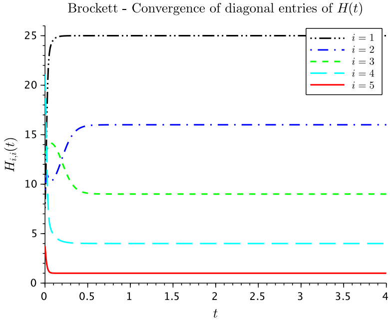
- The numerical solutions are computed using adaptive 4th-order Runge-Kutta scheme.

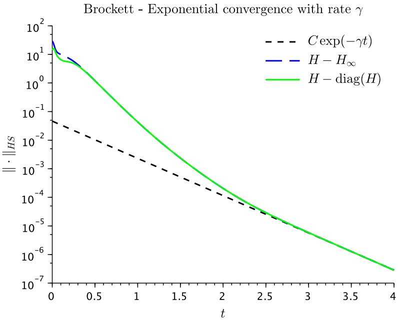
- Test the three flows presented before.

Brockett's choice

$H(t) \to H_\infty = \mathrm{diag}([25, 16, 9, 4, 1])$
Diagonal terms are sorted in a descending order, w.r.t to A

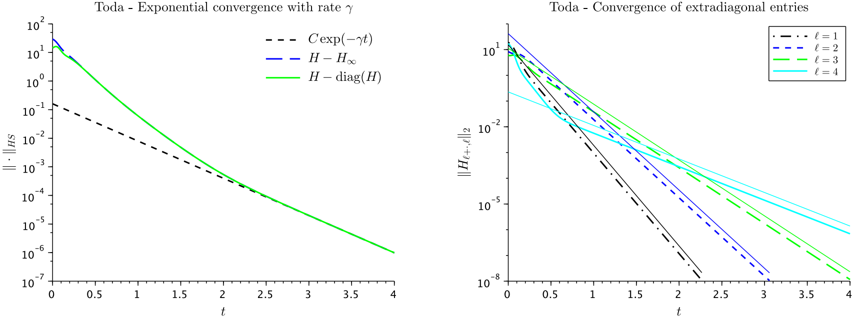
$$\mathrm{spec}\ \mathrm{d}F_{H_\infty} = \begin{pmatrix} 0 & -9 & -32 & -63 & -96 \\ -9 & 0 & -7 & -24 & -45 \\ -32 & -7 & 0 & -5 & -16 \\ -63 & -24 & -5 & 0 & -3 \\ -96 & -45 & -16 & -3 & 0 \end{pmatrix}.$$





Toda's choice

$$\text{spec } \mathrm{d}F_{H_\infty} = \begin{pmatrix} 0 & -9 & -16 & -21 & -24 \\ -9 & 0 & -7 & -12 & -15 \\ -16 & -7 & 0 & -5 & -8 \\ -21 & -12 & -5 & 0 & -3 \\ -24 & -15 & -8 & -3 & 0 \end{pmatrix}.$$

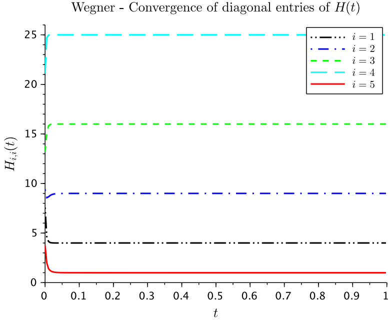


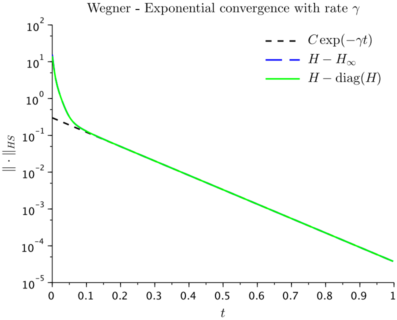
Wegner's choice

$H(t) \to H_\infty = \mathrm{diag}([4, 9, 16, 25, 1])$

$$\mathrm{d}F_{H_\infty}(E_{i,j}) = -(\lambda_i - \lambda_j)^2 E_{i,j} \,.$$

$$\mathrm{spec}\,\mathrm{d}F_{H_\infty} = \begin{pmatrix} 0 & -25 & -144 & -441 & -9 \\ -25 & 0 & -49 & -256 & -64 \\ -144 & -49 & 0 & -81 & -225 \\ -441 & -256 & -81 & 0 & -576 \\ -9 & -64 & -225 & -576 & 0 \end{pmatrix} .$$





Brockett's choice – other stable manifold

Other initial data : $\tilde{H}_0 = \tilde{Q} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 25 \end{pmatrix} \tilde{Q}^\star$, where $Q = \exp \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$.

- $\tilde{H}(t) = \begin{pmatrix} 1 & (0) \\ (0) & K(t) \end{pmatrix}$ with spec$(K(t)) = \{4, 9, 16, 25\}$.
- The (floating-point arithmetic) numerical method preserves this particular subspace
- $H(t) \to H_\infty =$ diag$([1, 25, 16, 9, 4])$ with exponential rate $\mathcal{O}(e^{-5t})$

$$\text{spec } \mathrm{d}F_{H_\infty} = \begin{pmatrix} 0 & 24 & 30 & 24 & 12 \\ 24 & 0 & -9 & -32 & -63 \\ 30 & -9 & 0 & -7 & -24 \\ 24 & -32 & -7 & 0 & -5 \\ 12 & -63 & -24 & -5 & 0 \end{pmatrix} .$$

Grub's up !

À table !

Ăn trưa được phục vụ !

午餐已送达