À propos d'existence globale dans des systèmes de réaction-diffusion

Michel Pierre

École Normale Supérieure de Rennes
and
Institut de Recherche Mathématique de Rennes

Journée Équipe AnaNum
24 mai 2022

Goal of the talk

- Story about the **global existence in time** of solutions to reaction-diffusion (RD) systems for which:
  - positivity of the solution is preserved for all time
  - conservation or dissipation or at least control of the total mass

Goal of the talk

- Story about the **global existence in time** of solutions to reaction-diffusion (RD) systems for which:
  - positivity of the solution is preserved for all time
  - conservation or dissipation or at least control of the total mass
- OLD ($>$'80) AND RECENT RESULTS- OPEN PROBLEMS

Goal of the talk

- Story about the **global existence in time** of solutions to reaction-diffusion (RD) systems for which:
  - positivity of the solution is preserved for all time
  - conservation or dissipation or at least control of the total mass
- OLD ($>$'80) AND RECENT RESULTS- OPEN PROBLEMS
- Lots of such systems in applications: chemical morphogenesis ('Brusselator'), Glycolosis, Gray-Scott models, combustion, Lotka-Volterra systems, epidemiology (SIR), reversible chemical reactions,...

Goal of the talk

- Story about the **global existence in time** of solutions to reaction-diffusion (RD) systems for which:
  - positivity of the solution is preserved for all time
  - conservation or dissipation or at least control of the total mass
- OLD (>'80) AND RECENT RESULTS- OPEN PROBLEMS
- Lots of such systems in applications: chemical morphogenesis ('Brusselator'), Glycolosis, Gray-Scott models, combustion, Lotka-Volterra systems, epidemiology (SIR), reversible chemical reactions,...
- The two properties provide an *a priori bound in L^1 for all time*.
  QUESTION: how does this help for global existence ???

Beginning of the story

- A question raised by R.H. Martin (Raleigh, North Carolina, USA) in 1980.

Beginning of the story

- A question raised by R.H. Martin (Raleigh, North Carolina, USA) in 1980.
- Does global existence of classical solutions hold for the following 2×2 reaction-diffusion system set on a good bounded domain $\Omega \subset \boldsymbol{R}^N$???

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^{\beta}, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^{\beta}, \\ u_i(0,\cdot) = u_i^0 \geq 0, \ i = 1, 2, \\ \text{good boundary conditions on } \partial\Omega, \end{cases}$$

where $d_1, d_2 \in (0, +\infty)$, $\beta \in [1, +\infty)$
and $u_i = u_i(t, x), t \in [0, T], x \in \Omega, \ i = 1, 2, \ T = +\infty$???.

Beginning of the story

- A question raised by R.H. Martin (Raleigh, North Carolina, USA) in 1980.
- Does global existence of classical solutions hold for the following 2×2 reaction-diffusion system set on a good bounded domain $\Omega \subset \boldsymbol{R}^N$???

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^{\beta}, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^{\beta}, \\ u_i(0,\cdot) = u_i^0 \geq 0, \ i = 1, 2, \\ \text{good boundary conditions on } \partial\Omega, \end{cases}$$

where $d_1, d_2 \in (0, +\infty)$, $\beta \in [1, +\infty)$
and $u_i = u_i(t,x), t \in [0, T], x \in \Omega, \ i = 1, 2, \ T = +\infty$???.

- Let us choose homogeneous Neumann boundary conditions.

Quid of global existence ???

- $d_1, d_2 \in (0, +\infty), \ \beta \in [1, +\infty)$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta, \\ u_i(0, \cdot) = u_i^0 \geq 0, \ i = 1, 2, \\ \partial_\nu u_i = 0, \ i = 1, 2 \ on \ \partial\Omega. \end{cases}$$

Quid of global existence ???

- $d_1, d_2 \in (0, +\infty)$, $\beta \in [1, +\infty)$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta, \\ u_i(0, \cdot) = u_i^0 \geq 0, \ i = 1, 2, \\ \partial_\nu u_i = 0, \ i = 1, 2 \ on \ \partial\Omega. \end{cases}$$

- Contains the O.D.E. case: $u_i = u_i(t)$:

$$\begin{cases} u_1'(t) = -u_1 u_2^\beta, \\ u_2'(t) = u_1 u_2^\beta, \\ u_i(0) = u_i^0 \in [0, +\infty) \ i = 1, 2. \end{cases}$$

Quid of global existence ???

- $d_1, d_2 \in (0, +\infty),\ \beta \in [1, +\infty)$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta, \\ u_i(0, \cdot) = u_i^0 \geq 0,\ i = 1, 2, \\ \partial_\nu u_i = 0,\ i = 1, 2 \ on\ \partial\Omega. \end{cases}$$

- Contains the O.D.E. case: $u_i = u_i(t)$:

$$\begin{cases} u_1'(t) = -u_1 u_2^\beta, \\ u_2'(t) = u_1 u_2^\beta, \\ u_i(0) = u_i^0 \in [0, +\infty)\ i = 1, 2. \end{cases}$$

- Local existence for the O.D.E. on a maximal interval $[0, T^*)$ by Cauchy-Lipschitz theorem, and the solution is nonnegative.

Quid of global existence ???

- $d_1, d_2 \in (0, +\infty)$, $\beta \in [1, +\infty)$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta, \\ u_i(0, \cdot) = u_i^0 \geq 0,\ i = 1, 2, \\ \partial_\nu u_i = 0,\ i = 1, 2 \text{ on } \partial\Omega. \end{cases}$$

- Contains the O.D.E. case: $u_i = u_i(t)$:

$$\begin{cases} u_1'(t) = -u_1 u_2^\beta, \\ u_2'(t) = u_1 u_2^\beta, \\ u_i(0) = u_i^0 \in [0, +\infty)\ i = 1, 2. \end{cases}$$

- Local existence for the O.D.E. on a maximal interval $[0, T^*)$ by Cauchy-Lipschitz theorem, and the solution is nonnegative.
- $(u_1 + u_2)'(t) = 0 \Rightarrow (u_1 + u_2)(t) = u_1^0 + u_2^0,\ \forall t \in [0, T^*)$

Quid of global existence ???

- $d_1, d_2 \in (0, +\infty)$, $\beta \in [1, +\infty)$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^{\beta}, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^{\beta}, \\ u_i(0, \cdot) = u_i^0 \geq 0, \ i = 1, 2, \\ \partial_\nu u_i = 0, \ i = 1, 2 \text{ on } \partial\Omega. \end{cases}$$

- Contains the O.D.E. case: $u_i = u_i(t)$:

$$\begin{cases} u_1'(t) = -u_1 u_2^{\beta}, \\ u_2'(t) = u_1 u_2^{\beta}, \\ u_i(0) = u_i^0 \in [0, +\infty) \ i = 1, 2. \end{cases}$$

- Local existence for the O.D.E. on a maximal interval $[0, T^*)$ by Cauchy-Lipschitz theorem, and the solution is nonnegative.
- $(u_1 + u_2)'(t) = 0 \Rightarrow (u_1 + u_2)(t) = u_1^0 + u_2^0, \ \forall t \in [0, T^*)$
- $\Rightarrow \ T^* = +\infty$!

Quid of global existence ???

- What about the full P.D.E. case?

$$
\begin{cases}
\partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^{\beta}, \\
\partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^{\beta}, \\
u_i(0,\cdot) = u_i^0 \geq 0, \; i = 1,2, \\
\partial_\nu u_i = 0, \; i = 1,2 \; on \; \partial\Omega.
\end{cases}
$$

Quid of global existence ???

- What about the full P.D.E. case?

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^{\beta}, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^{\beta}, \\ u_i(0,\cdot) = u_i^0 \geq 0, \ i = 1,2, \\ \partial_\nu u_i = 0, \ i = 1,2 \ on \ \partial\Omega. \end{cases}$$

- Local existence of classical solutions holds for $u_i^0 \in L^\infty(\Omega), i = 1, 2$ on a maximal interval $[0, T^*)$ by Cauchy-Lipschitz type theorem [[fixed point theorem in $L^\infty((0,T)\times\Omega)$]] and the solution is nonnegative as well.

Quid of global existence ???

- What about the full P.D.E. case?

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta, \\ u_i(0,\cdot) = u_i^0 \geq 0, \ i = 1,2, \\ \partial_\nu u_i = 0, \ i = 1,2 \text{ on } \partial\Omega. \end{cases}$$

- Local existence of classical solutions holds for $u_i^0 \in L^\infty(\Omega), i = 1,2$ on a maximal interval $[0, T^*)$ by Cauchy-Lipschitz type theorem [[fixed point theorem in $L^\infty((0,T) \times \Omega)$]] and the solution is nonnegative as well.
- Moreover $T^* = +\infty$ if the $L^\infty(\Omega)$-norm of $u_i(t)$ is bounded on $[0, T^*)$.

Quid of global existence ???

- What about the full P.D.E. case?

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^{\beta}, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^{\beta}, \\ u_i(0,\cdot) = u_i^0 \geq 0, \ i = 1,2, \\ \partial_\nu u_i = 0, \ i = 1,2 \ on \ \partial\Omega. \end{cases}$$

- Local existence of classical solutions holds for $u_i^0 \in L^\infty(\Omega), i = 1,2$ on a maximal interval $[0, T^*)$ by Cauchy-Lipschitz type theorem [[fixed point theorem in $L^\infty((0,T)\times\Omega)$]] and the solution is nonnegative as well.
- Moreover $T^* = +\infty$ if the $L^\infty(\Omega)$-norm of $u_i(t)$ is bounded on $[0, T^*)$.
- If $d_1 = d_2 = d$, $\partial_t(u_1+u_2) - d\Delta(u_1+u_2) = 0$

$$\Rightarrow \|(u_1+u_2)(t)\|_{L^\infty(\Omega)} \leq \|u_1^0 + u_2^0\|_{L^\infty(\Omega)}, \forall t \in [0, T^*)$$
$$\Rightarrow T^* = +\infty \ !$$

Quid of global existence ???

- What about the full P.D.E. case?

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta, \\ u_i(0,\cdot) = u_i^0 \geq 0, \ i = 1,2, \\ \partial_\nu u_i = 0, \ i = 1,2 \ on \ \partial\Omega. \end{cases}$$

- Local existence of classical solutions holds for $u_i^0 \in L^\infty(\Omega), i = 1,2$ on a maximal interval $[0, T^*)$ by Cauchy-Lipschitz type theorem [[fixed point theorem in $L^\infty((0,T) \times \Omega)$]] and the solution is nonnegative as well.
- Moreover $T^* = +\infty$ if the $L^\infty(\Omega)$-norm of $u_i(t)$ is bounded on $[0, T^*)$.
- If $d_1 = d_2 = d$, $\partial_t(u_1 + u_2) - d\Delta(u_1 + u_2) = 0$

$$\Rightarrow \|(u_1 + u_2)(t)\|_{L^\infty(\Omega)} \leq \|u_1^0 + u_2^0\|_{L^\infty(\Omega)}, \forall t \in [0, T^*)$$
$$\Rightarrow T^* = +\infty \ !$$

- What happens when $d_1 \neq d_2$???.

Quid of global existence ? An $L^1(\Omega)$-estimate

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0, \\ \partial_\nu u_i = 0 \;\; on\; \partial\Omega, \;\; i = 1, 2. \end{cases}$$

- Using $\int_\Omega \Delta u_i = \int_{\partial\Omega} \partial_\nu u_i = 0$, we have

$$\frac{d}{dt} \int_\Omega (u_1 + u_2)(t) = \int_\Omega \partial_t (u_1 + u_2) = 0$$

$$\Rightarrow \quad \int_\Omega u_1(t) + u_2(t) = \int_\Omega u_1^o + u_2^o$$

Quid of global existence ? An $L^1(\Omega)$-estimate

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0, \\ \partial_\nu u_i = 0 \ \ on\ \partial\Omega, \ \ i = 1, 2. \end{cases}$$

- Using $\int_\Omega \Delta u_i = \int_{\partial\Omega} \partial_\nu u_i = 0$, we have

$$\frac{d}{dt} \int_\Omega (u_1 + u_2)(t) = \int_\Omega \partial_t (u_1 + u_2) = 0$$

$$\Rightarrow \quad \int_\Omega u_1(t) + u_2(t) = \int_\Omega u_1^o + u_2^o$$

- $\Rightarrow \quad L^1(\Omega)$-bound, uniform in time $[t \in [0, T^*)]$!

Quid of global existence ? An $L^1(\Omega)$-estimate

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0, \\ \partial_\nu u_i = 0 \;\; on\ \partial\Omega, \;\; i = 1, 2. \end{cases}$$

- Using $\int_\Omega \Delta u_i = \int_{\partial\Omega} \partial_\nu u_i = 0$, we have

$$\frac{d}{dt} \int_\Omega (u_1 + u_2)(t) = \int_\Omega \partial_t (u_1 + u_2) = 0$$

$$\Rightarrow \quad \int_\Omega u_1(t) + u_2(t) = \int_\Omega u_1^o + u_2^o$$

- $\Rightarrow$ $L^1(\Omega)$-bound, uniform in time $[t \in [0, T^*)]$!
- How does this help for global existence?

Same question for the family of systems with the two main properties **(P)+(M)** which yield the same estimates

$$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m), \\ \partial_\nu u_i = 0, \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0, \end{cases}$$

$d_i \in (0, +\infty)$, $f_i : [0,\infty)^m \to \boldsymbol{R}$ locally Lipschitz continuous,

- **(P):** Positivity (nonnegativity) is preserved

Same question for the family of systems with the two main properties **(P)+(M)** which yield the same estimates

$$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m), \\ \partial_\nu u_i = 0, \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0, \end{cases}$$

$d_i \in (0, +\infty)$, $f_i : [0, \infty)^m \to \boldsymbol{R}$ locally Lipschitz continuous,

- **(P):** Positivity (nonnegativity) is preserved
- **(M):** $\sum_{1 \leq i \leq m} f_i \leq 0$

Same question for the family of systems with the two main properties **(P)+(M)** which yield the same estimates

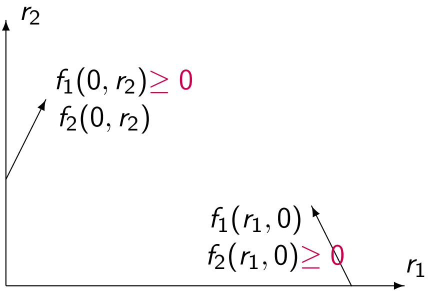
$$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m), \\ \partial_\nu u_i = 0, \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0, \end{cases}$$

$d_i \in (0, +\infty)$, $f_i : [0, \infty)^m \to \boldsymbol{R}$ locally Lipschitz continuous,

- **(P):** Positivity (nonnegativity) is preserved
- **(M):** $\sum_{1 \leq i \leq m} f_i \leq 0$
- or more generally
  (M') $\forall r \in [0, \infty[^m, \; \sum_{1 \leq i \leq m} a_i f_i(r) \leq C[1 + \sum_{1 \leq i \leq m} r_i]$
  for some $a_i > 0$

$$
(S)\begin{cases}
\forall i = 1, ..., m & \\
\partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T := (0, T) \times \Omega, \\
\partial_\nu u_i = 0 & \text{on } \Sigma_T := (0, T) \times \partial\Omega, \\
u_i(0, \cdot) = u_i^0(\cdot) \geq 0. &
\end{cases}
$$

- **(P) Preservation of Positivity:** $\forall i = 1, ..., m$
  $\forall r = (r_1, ..., r_m) \in [0, \infty[^m,\ f_i(r_1, ..., r_{i-1}, 0, r_{i+1}, ..., r_m) \geq 0,$
  $=$ " quasi-positivity " of $f = (f_i)_{1 \leq i \leq m}$.



- $$(S)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$

- **(P) Preservation of Positivity:** $\forall i = 1, ..., m$
  $\forall r \in [0, +\infty[^m, \quad f_i(r_1, ..., r_{i-1}, 0, r_{i+1}, ..., r_m) \geq 0.$
- **(M):** $\sum_{1 \leq i \leq m} f_i(r_1, ..., r_m) \leq 0 \Rightarrow$ **'Control of the Total Mass':**

$$\forall t \geq 0, \quad \int_\Omega \sum_{1 \leq i \leq r} u_i(t, x)dx \leq \int_\Omega \sum_{1 \leq i \leq r} u_i^0(x)dx$$

Add up, integrate on Ω, use $\int_\Omega \Delta u_i = \int_{\partial\Omega} \partial_\nu u_i = 0$:

$$\int_\Omega \partial_t [\sum u_i(t)]dx = \int_\Omega \sum_i f_i(u)dx \leq 0.$$

- $$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & in\ Q_T \\ \partial_\nu u_i = 0 & on\ \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

- **(P) Preservation of Positivity:** $\forall i = 1, ..., m$ $\forall r \in [0, +\infty[^m, \quad f_i(r_1, ..., r_{i-1}, 0, r_{i+1}, ..., r_m) \geq 0.$
- **(M):** $\sum_{1 \leq i \leq m} f_i(r_1, ..., r_m) \leq 0 \Rightarrow$ **'Control of the Total Mass':**

$$\forall t \geq 0, \quad \int_\Omega \sum_{1 \leq i \leq r} u_i(t, x) dx \leq \int_\Omega \sum_{1 \leq i \leq r} u_i^0(x) dx$$

Add up, integrate on Ω, use $\int_\Omega \Delta u_i = \int_{\partial\Omega} \partial_\nu u_i = 0$:

$$\int_\Omega \partial_t [\sum u_i(t)] dx = \int_\Omega \sum_i f_i(u) dx \leq 0.$$

- $\Rightarrow L^1(\Omega)$- a priori estimates, uniform in time.

- $$(S)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$
- **(P) Preservation of Positivity:** $\forall i = 1, ..., m$ $\forall r \in [0, +\infty[^m, \quad f_i(r_1, ..., r_{i-1}, 0, r_{i+1}, ..., r_m) \geq 0.$
- **(M):** $\sum_{1\leq i\leq m} f_i(r_1, ..., r_m) \leq 0 \Rightarrow$ **'Control of the Total Mass':**

$$\forall t \geq 0, \quad \int_\Omega \sum_{1\leq i\leq r} u_i(t,x)dx \leq \int_\Omega \sum_{1\leq i\leq r} u_i^0(x)dx$$

Add up, integrate on Ω, use $\int_\Omega \Delta u_i = \int_{\partial\Omega} \partial_\nu u_i = 0$:

$$\int_\Omega \partial_t[\sum u_i(t)]dx = \int_\Omega \sum_i f_i(u)dx \leq 0.$$

- $\Rightarrow L^1(\Omega)$- a priori estimates, uniform in time.
- Remark: L^1-bound for all time with **(M')**

QUESTION:

What about Global Existence of solutions

under assumption **(P)+(M)** *??*

or more generally **(P)+ (M')** *??*

Several approaches and techniques

- L^∞-approach: local existence
- An L^p-approach
- Blow up may occur...
- An L^1-approach
- $L\,Log\,L$ may also be involved
- A surprising L^2-estimate
- And more about quadratic systems
-
- ...based on various properties of the Heat Operator and of diffusion operators with (only) bounded coefficients
- + OPEN PROBLEMS

Local existence in L^∞ for systems

$$(S)\left\{\begin{array}{ll} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \textit{in } Q_T \\ \partial_\nu u_i = 0 & \textit{on } \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. & \end{array}\right. :$$

- **Theorem** (*à la Cauchy-Lipschitz dans* L^∞).
  Let $u^0 = (u_i^0)_{1\leq i\leq m} \in L^\infty(\Omega)^{+m}$. Then, there exist a maximum time $T^* > 0$ and $u = (u_1, ..., u_m)$ unique classical nonnegative solution of (S) on $[0, T^*)$. Moreover,

$$\sup_{t\in[0,T^*)} \left\{\max_i \|u_i(t)\|_{L^\infty(\Omega)}\right\} < +\infty \Rightarrow [T^* + \infty] .$$

Local existence in L^∞ for systems

$$(S)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \textit{in } Q_T \\ \partial_\nu u_i = 0 & \textit{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. & \end{cases} :$$

- **Theorem** (*à la Cauchy-Lipschitz dans* L^∞).
  Let $u^0 = (u_i^0)_{1\leq i\leq m} \in L^\infty(\Omega)^{+m}$. Then, there exist a maximum time $T^* > 0$ and $u = (u_1, ..., u_m)$ unique classical nonnegative solution of (S) on $[0, T^*)$. Moreover,

$$\sup_{t\in[0,T^*)} \left\{ \max_i \|u_i(t)\|_{L^\infty(\Omega)} \right\} < +\infty \Rightarrow [T^* + \infty] \, .$$

- **Corollary.** If $d_i = d$ for all $i = 1, ..., m$, then $T^* = +\infty$.

Local existence in L^∞ for systems

$$(S)\begin{cases} \forall i=1,...,m & \\ \partial_t u_i - d_i\Delta u_i = f_i(u_1,u_2,...,u_m) & \textit{in } Q_T \\ \partial_\nu u_i = 0 & \textit{on } \Sigma_T \\ u_i(0,\cdot)=u_i^0(\cdot)\geq 0. & \end{cases} :$$

- **Theorem** (*à la Cauchy-Lipschitz dans* L^∞).
Let $u^0=(u_i^0)_{1\leq i\leq m}\in L^\infty(\Omega)^{+m}$. Then, there exist a maximum time $T^*>0$ and $u=(u_1,...,u_m)$ unique classical nonnegative solution of (S) on $[0,T^*)$. Moreover,

$$\sup_{t\in[0,T^*)}\left\{\max_i \|u_i(t)\|_{L^\infty(\Omega)}\right\}<+\infty \Rightarrow [T^*+\infty].$$

- **Corollary.** If $d_i=d$ for all $i=1,...,m$, then $T^*=+\infty$.
- **Proof:** $\partial_t(\sum_i u_i)-d\,\Delta(\sum_i u_i)\leq 0$.

$$\Rightarrow \|\sum_i u_i(t)\|_{L^\infty(\Omega)}\leq \|\sum_i u_{i0}\|_{L^\infty(\Omega)}.$$

The L^p-approach

- Recall the R.H. Martin's problem ($\beta \in [1, +\infty)$)

$$(S)\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \;\; (\leq 0) \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ \partial_\nu u_i = 0 \; on \; \partial\Omega, \;\; i = 1, 2. \end{cases}$$

The L^p-approach

- Recall the R.H. Martin's problem ($\beta \in [1, +\infty)$)

$$(S) \begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \ \ (\leq 0) \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ \partial_\nu u_i = 0 \ on\ \partial\Omega, \ \ i = 1, 2. \end{cases}$$

- By maximum principle $\|u_1(t)\|_{L^\infty(\Omega)} \leq \|u_1^0\|_{L^\infty(\Omega)}$.

The L^p-approach

- Recall the R.H. Martin's problem ($\beta \in [1, +\infty)$)

$$(S)\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \ \ (\leq 0) \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ \partial_\nu u_i = 0 \ on \ \partial\Omega, \ \ i = 1, 2. \end{cases}$$

- By maximum principle $\|u_1(t)\|_{L^\infty(\Omega)} \leq \|u_1^0\|_{L^\infty(\Omega)}$.
- We have : $\partial_t u_2 - d_2 \Delta u_2 = -(\partial_t u_1 - d_1 \Delta u_1)$.

The L^p-approach

- Recall the R.H. Martin's problem ($\beta \in [1, +\infty)$)

$$(S)\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \ \ (\leq 0) \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ \partial_\nu u_i = 0 \ on \ \partial\Omega, \ \ i = 1, 2. \end{cases}$$

- By maximum principle $\|u_1(t)\|_{L^\infty(\Omega)} \leq \|u_1^0\|_{L^\infty(\Omega)}$.
- We have : $\partial_t u_2 - d_2 \Delta u_2 = -(\partial_t u_1 - d_1 \Delta u_1)$.
- Or : " $u_2 = (\partial_t - d_2\Delta)^{-1} (-\partial_t + d_1 \Delta)\, u_1$" $[=: \mathcal{A} u_1]$

The L^p-approach

- Recall the R.H. Martin's problem ($\beta \in [1, +\infty)$)

$$(S) \begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \ \ (\leq 0) \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ \partial_\nu u_i = 0 \ on \ \partial\Omega, \ \ i = 1, 2. \end{cases}$$

- By maximum principle $\|u_1(t)\|_{L^\infty(\Omega)} \leq \|u_1^0\|_{L^\infty(\Omega)}$.
- We have : $\partial_t u_2 - d_2 \Delta u_2 = -(\partial_t u_1 - d_1 \Delta u_1)$.
- Or : " $u_2 = (\partial_t - d_2 \Delta)^{-1} (-\partial_t + d_1 \Delta) u_1$" $[=: \mathcal{A} u_1]$
- L^p-**Main Lemma**: the operator $\mathcal{A}$ is continuous from $L^p(Q_T)$ into itself for all $p \in (1, +\infty)$ and all $T > 0$.

$$\Rightarrow \|u_2\|_{L^p(Q_{T^*})} < +\infty, \forall p < +\infty.$$

The L^p-approach

- Recall the R.H. Martin's problem ($\beta \in [1, +\infty)$)

$$(S) \begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \ \ (\leq 0) \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ \partial_\nu u_i = 0 \ on \ \partial\Omega, \ \ i = 1, 2. \end{cases}$$

- By maximum principle $\|u_1(t)\|_{L^\infty(\Omega)} \leq \|u_1^0\|_{L^\infty(\Omega)}$.
- We have : $\partial_t u_2 - d_2 \Delta u_2 = -(\partial_t u_1 - d_1 \Delta u_1)$.
- Or : "$u_2 = (\partial_t - d_2 \Delta)^{-1} (-\partial_t + d_1 \Delta) \, u_1$" $[=: \mathcal{A} u_1]$
- **L^p-Main Lemma**: the operator $\mathcal{A}$ is continuous from $L^p(Q_T)$ into itself for all $p \in (1, +\infty)$ and all $T > 0$.

$$\Rightarrow \|u_2\|_{L^p(Q_{T^*})} < +\infty, \forall p < +\infty.$$

- $\Rightarrow \|u_1 u_2^\beta\|_{L^q(Q_{T^*})} < +\infty, \forall q < +\infty,$

The L^p-approach

- Recall the R.H. Martin's problem ($\beta \in [1, +\infty)$)

$$(S) \begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \ \ (\leq 0) \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ \partial_\nu u_i = 0 \ on \ \partial\Omega, \ \ i = 1, 2. \end{cases}$$

- By maximum principle $\|u_1(t)\|_{L^\infty(\Omega)} \leq \|u_1^0\|_{L^\infty(\Omega)}$.
- We have : $\partial_t u_2 - d_2 \Delta u_2 = -(\partial_t u_1 - d_1 \Delta u_1)$.
- Or : "$u_2 = (\partial_t - d_2 \Delta)^{-1} (-\partial_t + d_1 \Delta) u_1$" $[=: \mathcal{A} u_1]$
- **L^p-Main Lemma**: the operator $\mathcal{A}$ is continuous from $L^p(Q_T)$ into itself for all $p \in (1, +\infty)$ and all $T > 0$.

$$\Rightarrow \|u_2\|_{L^p(Q_{T^*})} < +\infty, \forall p < +\infty.$$

- $\Rightarrow \|u_1 u_2^\beta\|_{L^q(Q_{T^*})} < +\infty, \forall q < +\infty,$
- Choose $q > (N+1)/2$ and use the equation in u_2 to deduce:

The L^p-approach

- Recall the R.H. Martin's problem ($\beta \in [1, +\infty)$)

$$(S) \begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 u_2^\beta \ \ (\leq 0) \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 u_2^\beta \\ \partial_\nu u_i = 0 \ on \ \partial\Omega, \ \ i = 1, 2. \end{cases}$$

- By maximum principle $\|u_1(t)\|_{L^\infty(\Omega)} \leq \|u_1^0\|_{L^\infty(\Omega)}$.
- We have : $\partial_t u_2 - d_2 \Delta u_2 = -(\partial_t u_1 - d_1 \Delta u_1)$.
- Or : " $u_2 = (\partial_t - d_2 \Delta)^{-1} (-\partial_t + d_1 \Delta) u_1$" $[=: \mathcal{A} u_1]$
- L^p**-Main Lemma**: the operator $\mathcal{A}$ is continuous from $L^p(Q_T)$ into itself for all $p \in (1, +\infty)$ and all $T > 0$.

$$\Rightarrow \|u_2\|_{L^p(Q_{T^*})} < +\infty, \forall p < +\infty.$$

- $\Rightarrow \|u_1 u_2^\beta\|_{L^q(Q_{T^*})} < +\infty, \forall q < +\infty,$
- Choose $q > (N+1)/2$ and use the equation in u_2 to deduce:
- $\Rightarrow \|u_2\|_{L^\infty(Q_{T^*})} < +\infty$ *and* $T^* = +\infty$!

The L^p-Main Lemma is a dual statement of the maximal $L^{p'}$-regularity

- More generally, [S. Hollis, M.P., R.H. Martin '87]

$$\partial_t u_2 - d_2 \Delta u_2 \leq a \partial_t u_1 + b \Delta u_1, \quad u_2 \geq 0 + B.C., a, b \in \mathbb{R},$$

implies the existence of $C = C(p, T, \Omega, u_i^0, a, b)$ such that:

$$\forall p \in (1, \infty), \ \|u_2\|_{L^p(Q_T)} \leq C \left[1 + \|u_1\|_{L^p(Q_T)}\right].$$

The L^p-Main Lemma is a dual statement of the maximal $L^{p'}$-regularity

- More generally, [S. Hollis, M.P., R.H. Martin '87]

$$\partial_t u_2 - d_2 \Delta u_2 \leq a \partial_t u_1 + b \Delta u_1, \quad u_2 \geq 0 + B.C., a, b \in \boldsymbol{R},$$

implies the existence of $C = C(p, T, \Omega, u_i^0, a, b)$ such that:

$$\forall p \in (1, \infty), \ \|u_2\|_{L^p(Q_T)} \leq C \left[1 + \|u_1\|_{L^p(Q_T)}\right].$$

- Solve the dual problem

$$\begin{cases} -(\partial_t \psi + d_2 \Delta \psi) = \Theta \ \in \ C_0^\infty(Q_T), \Theta \geq 0, \\ \psi(T) = 0, \quad \partial_\nu \psi = 0 \ on \ \Sigma_T. \end{cases}$$

The L^p-Main Lemma is a dual statement of the maximal $L^{p'}$-regularity

- More generally, [S. Hollis, M.P., R.H. Martin '87]

$$\partial_t u_2 - d_2 \Delta u_2 \leq a \partial_t u_1 + b \Delta u_1, \quad u_2 \geq 0 + B.C., a, b \in \boldsymbol{R},$$

implies the existence of $C = C(p, T, \Omega, u_i^0, a, b)$ such that:

$$\forall p \in (1, \infty), \; \|u_2\|_{L^p(Q_T)} \leq C \left[1 + \|u_1\|_{L^p(Q_T)}\right].$$

- Solve the dual problem

$$\begin{cases} -(\partial_t \psi + d_2 \Delta \psi) = \Theta \; \in \; C_0^\infty(Q_T), \Theta \geq 0, \\ \psi(T) = 0, \quad \partial_\nu \psi = 0 \; on \; \Sigma_T. \end{cases}$$

- Multiplying the inequality in u_2 by $\psi \geq 0$ leads to:

$$\int_{Q_T} u_2 \Theta \leq \int_\Omega (-a u_1^0 + u_2^0) \psi(0) + a \int_{Q_T} u_1 \Theta + (a d_2 + b) \int_{Q_T} u_1 \Delta \psi.$$

The L^p-Main Lemma is a dual statement of the maximal $L^{p'}$-regularity

- More generally, [S. Hollis, M.P., R.H. Martin '87]

$$\partial_t u_2 - d_2\Delta u_2 \leq a\partial_t u_1 + b\Delta u_1, \quad u_2 \geq 0 + B.C., a, b \in \boldsymbol{R},$$

  implies the existence of $C = C(p, T, \Omega, u_i^0, a, b)$ such that:

$$\forall p \in (1,\infty), \; \|u_2\|_{L^p(Q_T)} \leq C\,[1 + \|u_1\|_{L^p(Q_T)}].$$

- Solve the dual problem

$$\begin{cases} -(\partial_t\psi + d_2\Delta\psi) = \Theta \; \in \; C_0^\infty(Q_T), \Theta \geq 0, \\ \psi(T) = 0, \quad \partial_\nu\psi = 0 \; on \; \Sigma_T. \end{cases}$$

- Multiplying the inequality in u_2 by $\psi \geq 0$ leads to:

$$\int_{Q_T} u_2\Theta \leq \int_\Omega (-a u_1^0 + u_2^0)\psi(0) + a\int_{Q_T} u_1\Theta + (ad_2 + b)\int_{Q_T} u_1\Delta\psi.$$

- By the $L^{p'}$-maximal regularity theory for the heat operator

$$\|\Delta\psi\|_{L^{p'}(Q_T)} + \|\psi(0)\|_{L^{p'}(\Omega)} \leq C\|\Theta\|_{L^{p'}(Q_T)}.$$

The L^p-Main Lemma is a dual statement of the maximal $L^{p'}$-regularity

- More generally, [S. Hollis, M.P., R.H. Martin '87]

$$\partial_t u_2 - d_2 \Delta u_2 \leq a \partial_t u_1 + b \Delta u_1, \quad u_2 \geq 0 + B.C., a, b \in \mathbb{R},$$

implies the existence of $C = C(p, T, \Omega, u_i^0, a, b)$ such that:

$$\forall p \in (1, \infty), \ \|u_2\|_{L^p(Q_T)} \leq C \left[1 + \|u_1\|_{L^p(Q_T)}\right].$$

- Solve the dual problem

$$\begin{cases} -(\partial_t \psi + d_2 \Delta \psi) = \Theta \ \in \ C_0^\infty(Q_T), \Theta \geq 0, \\ \psi(T) = 0, \quad \partial_\nu \psi = 0 \ on \ \Sigma_T. \end{cases}$$

- Multiplying the inequality in u_2 by $\psi \geq 0$ leads to:

$$\int_{Q_T} u_2 \Theta \leq \int_\Omega (-a u_1^0 + u_2^0) \psi(0) + a \int_{Q_T} u_1 \Theta + (a d_2 + b) \int_{Q_T} u_1 \Delta \psi.$$

- By the $L^{p'}$-maximal regularity theory for the heat operator

$$\|\Delta \psi\|_{L^{p'}(Q_T)} + \|\psi(0)\|_{L^{p'}(\Omega)} \leq C \|\Theta\|_{L^{p'}(Q_T)}.$$

- $\Rightarrow \left| \int_{Q_T} u_2 \Theta \right| \leq C[1 + \|u_1\|_{L^p(Q_T)}] \|\Theta\|_{L^{p'}(Q_T)} \Rightarrow L^p(Q_T)$-estimate on u_2 by duality.

Extensions and limits of the L^p-approach

$$\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$

- The same approach provides global existence for the general system when a triangular structure holds like

$$f_1 \leq 0, \quad f_1 + f_2 \leq 0, f_1 + f_2 + f_3 \leq 0, ...$$

in which case we have, with $Q = Q_{T^*}$ and for all $p \in (1, \infty)$

$$\|u_1\|_{L^\infty(Q)} < \infty, \ \|u_2\|_{L^p(Q)} \leq C\|u_1\|_{L^p(Q)}$$
$$\|u_3\|_{L^p(Q)} \leq C[\|u_1\|_{L^p(Q)} + \|u_2\|_{L^p(Q)}], ...$$

Extensions and limits of the L^p-approach

$$\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

- The same approach provides global existence for the general system when a triangular structure holds like

$$f_1 \leq 0, \quad f_1 + f_2 \leq 0, f_1 + f_2 + f_3 \leq 0, ...$$

in which case we have, with $Q = Q_{T^*}$ and for all $p \in (1, \infty)$

$$\|u_1\|_{L^\infty(Q)} < \infty, \ \|u_2\|_{L^p(Q)} \leq C\|u_1\|_{L^p(Q)}$$

$$\|u_3\|_{L^p(Q)} \leq C[\|u_1\|_{L^p(Q)} + \|u_2\|_{L^p(Q)}], ...$$

- [J. Morgan, W. Fitzgibbon, et al. '89] More generally it applies to $m \times m$ systems if there exists a triangular invertible matrix Q with nonnegative entries such that

$$\forall r \in [0, \infty)^m, \ Q\,f(r) \leq [1 + \sum_{1 \leq i \leq m} r_i]\mathrm{b},$$

for some $\mathrm{b} \in \mathbb{R}^m, f = (f_1, ..., f_m)^t$ with at most polynomial growth.

Extension with advection and anisotropic diffusion

$$(S)\begin{cases} \partial_t u_i - div\,(D_i(t,x)\nabla u_i + V_i(t,x)u_i) = f_i(t,x,u), \\ (D_i(t,x)\nabla u_i + V_i(t,x)u_i)\cdot\nu = 0 \text{ on } \partial\Omega, \\ u_i(0,\cdot) = u_i^0 \geq 0, \\ D_i = \left[d_i^{lk}\right]_{1\leq k,l\leq N} \text{ symmetric elliptic}, \quad V_i \in \boldsymbol{R}^N. \end{cases}$$

- **Theorem.** [D. Bothe, A. Fischer, M.P., G. Rolland, '2016] Assume that $f = (f_1, ..., f_m)$ satisfies **(P), (M')**, the triangular structure and with growth at most polynomial. Assume also that,

$$V_i, \nabla d_i^{lk} \in L^\infty(0,T; L^r(\Omega)) \text{ for some } r > \max\{2, N\},$$

$$d_i^{lk} \in C(\overline{Q_T}), \quad \forall\, T > 0.$$

Then, there are global bounded solutions for (S).

Extension with advection and anisotropic diffusion

$$(S)\begin{cases} \partial_t u_i - div\left(D_i(t,x)\nabla u_i + V_i(t,x)u_i\right) = f_i(t,x,u), \\ \left(D_i(t,x)\nabla u_i + V_i(t,x)u_i\right)\cdot\nu = 0 \text{ on } \partial\Omega, \\ u_i(0,\cdot) = u_i^0 \geq 0, \\ D_i = \left[d_i^{lk}\right]_{1\leq k,l\leq N} \text{ symmetric elliptic}, \quad V_i \in \boldsymbol{R}^N. \end{cases}$$

- **Theorem.** [D. Bothe, A. Fischer, M.P., G. Rolland, '2016] Assume that $f = (f_1, ..., f_m)$ satisfies **(P), (M')**, the triangular structure and with growth at most polynomial. Assume also that,

$$V_i, \nabla d_i^{lk} \in L^\infty\left(0, T; L^r(\Omega)\right) \text{ for some } r > \max\{2, N\},$$

$$d_i^{lk} \in C(\overline{Q_T}), \quad \forall\, T > 0.$$

Then, there are global bounded solutions for (S).

- The assumptions are so that $L^{p'}$-regularity theory holds for each dual problem [H. Amann, R. Denk-M. Hieber-J. Prüss, '05]

$$-\left[\partial_t\Psi + div\left(D_i(t,x)\nabla\Psi\right)\right] + V_i(t,x)\cdot\nabla\Psi = \Theta \in C_0^\infty((\tau,\tau+\delta),$$

$$D_i(\tau,x)\nabla\Psi\cdot\nu = \theta \in C^\infty\left((\tau,\tau+\delta)\times\partial\Omega\right),$$

where δ is small.

Application to the case of close d_i's

$$(S)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. & \end{cases} :$$

- We may write for $\underline{d} := \min_i d_i, \overline{d} := \max_i d_i$

$$\begin{cases} \partial_t(\sum_i u_i) - \underline{d}\Delta(\sum_i u_i) & = \sum_i (d_i - \underline{d})\Delta u_i + \sum_i f_i \\ & \leq \Delta\left(\sum_i (d_i - \underline{d}) u_i\right). \end{cases}$$

Application to the case of close d_i's

$$(S)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. & \end{cases} :$$

- We may write for $\underline{d} := \min_i d_i, \overline{d} := \max_i d_i$

$$\begin{cases} \partial_t(\sum_i u_i) - \underline{d}\Delta(\sum_i u_i) & = \sum_i (d_i - \underline{d})\Delta u_i + \sum_i f_i \\ & \leq \Delta\left(\sum_i (d_i - \underline{d}) u_i\right). \end{cases}$$

- We deduce from the L^p-Main Lemma that $\forall p \in (1, +\infty)$

$$\begin{cases} \|\sum_i u_i\|_{L^p(Q_T)} & \leq C\left[1 + \|\sum_i (d_i - \underline{d}) u_i\|_{L^p(Q_T)}\right] \\ & \leq C\left[1 + (\overline{d} - \underline{d})\|\sum_i u_i\|_{L^p(Q_T)}\right]. \end{cases}$$

Application to the case of close d_i's

$$(S)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & in\ Q_T \\ \partial_\nu u_i = 0 & on\ \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. & \end{cases} :$$

- We may write for $\underline{d} := \min_i d_i, \overline{d} := \max_i d_i$

$$\begin{cases} \partial_t(\sum_i u_i) - \underline{d}\Delta(\sum_i u_i) & = \sum_i (d_i - \underline{d})\Delta u_i + \sum_i f_i \\ & \leq \Delta\left(\sum_i (d_i - \underline{d}) u_i\right). \end{cases}$$

- We deduce from the L^p-Main Lemma that $\forall p \in (1, +\infty)$

$$\begin{cases} \| \sum_i u_i \|_{L^p(Q_T)} & \leq C\left[1 + \| \sum_i (d_i - \underline{d}) u_i \|_{L^p(Q_T)}\right] \\ & \leq C\left[1 + (\overline{d} - \underline{d}) \| \sum_i u_i \|_{L^p(Q_T)}\right]. \end{cases}$$

- If the $d_i' s$ are close enough so that $C(\overline{d} - \underline{d}) < 1$, then

$$\| \sum_i u_i \|_{L^p(Q_{T^*})} \leq C[1 - C(\overline{d} - \underline{d})]^{-1} < +\infty.$$

Application to the case of close d_i's

$$(S)\begin{cases} \forall i = 1, \ldots, m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, \ldots, u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. & \end{cases} :$$

- We may write for $\underline{d} := \min_i d_i, \overline{d} := \max_i d_i$

$$\begin{cases} \partial_t(\sum_i u_i) - \underline{d}\Delta(\sum_i u_i) & = \sum_i (d_i - \underline{d})\Delta u_i + \sum_i f_i \\ & \leq \Delta\left(\sum_i (d_i - \underline{d}) u_i\right). \end{cases}$$

- We deduce from the L^p-Main Lemma that $\forall p \in (1, +\infty)$

$$\begin{cases} \|\sum_i u_i\|_{L^p(Q_T)} & \leq C\left[1 + \|\sum_i (d_i - \underline{d}) u_i\|_{L^p(Q_T)}\right] \\ & \leq C\left[1 + (\overline{d} - \underline{d})\|\sum_i u_i\|_{L^p(Q_T)}\right]. \end{cases}$$

- If the $d_i' s$ are close enough so that $C(\overline{d} - \underline{d}) < 1$, then

$$\|\sum_i u_i\|_{L^p(Q_{T^*})} \leq C[1 - C(\overline{d} - \underline{d})]^{-1} < +\infty.$$

- Whence global existence if, moreover, f_i at most polynomial !

Extensions and limits of the L^p-approach

- L^p-approach does not apply to

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 e^{u_2^2} \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 e^{u_2^2} \end{cases}$$

Extensions and limits of the L^p-approach

- L^p-approach does not apply to

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 e^{u_2^2} \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 e^{u_2^2} \end{cases}$$

- neither to the system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = u_1^3 u_2^2 - u_1^2 u_2^3 \\ \partial_t u_2 - d_2 \Delta u_2 = u_1^2 u_2^3 - u_1^3 u_2^2 \end{cases}$$

Extensions and limits of the L^p-approach

- L^p-approach does not apply to

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 e^{u_2^2} \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 e^{u_2^2} \end{cases}$$

- neither to the system

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = u_1^3 u_2^2 - u_1^2 u_2^3 \\ \partial_t u_2 - d_2 \Delta u_2 = u_1^2 u_2^3 - u_1^3 u_2^2 \end{cases}$$

- and even not to the "better" system with $\lambda \in (0,1)$

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = \lambda u_1^3 u_2^2 - u_1^2 u_2^3 \; [=: f_1(u)] \\ \partial_t u_2 - d_2 \Delta u_2 = u_1^2 u_2^3 - u_1^3 u_2^2 \; [=: f_2(u)] \end{cases}$$

where : $f_1(u) + f_2(u) \leq 0$,
and also : $f_1(u) + \lambda f_2(u) \leq 0$

Finite time L^∞-blow up may appear with **(M)+(P)** !

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ + \text{ various "good" boundary conditions} \end{cases}$$

- Theorem: [D. Schmitt-MP, 90'] One can find 'polynomial' nonlinearities f, g satisfying **(P)** and

$$\textbf{(M)}\ f + g \leq 0, \ \textit{and also}: \exists \lambda \in [0, 1[, f + \lambda g \leq 0,$$

for which $T^* < +\infty$ with

$$\lim_{t \to T^*} \|u_1(t)\|_{L^\infty(\Omega)} = \lim_{t \to T^*} \|u_2(t)\|_{L^\infty(\Omega)} = +\infty.$$

Finite time L^∞-blow up may appear with **(M)+(P)** !

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ + \text{ various "good" boundary conditions} \end{cases}$$

- Theorem: [D. Schmitt-MP, 90'] One can find 'polynomial' nonlinearities f, g satisfying **(P)** and

$$\textbf{(M)}\ f + g \leq 0,\ \textit{and also}: \exists \lambda \in [0,1[,\ f + \lambda g \leq 0,$$

for which $T^* < +\infty$ with

$$\lim_{t \to T^*} \|u_1(t)\|_{L^\infty(\Omega)} = \lim_{t \to T^*} \|u_2(t)\|_{L^\infty(\Omega)} = +\infty.$$

- Blow up may appear even in space dimension $N = 1$ (with high degree polynomial nonlinearities)

Finite time L^∞-blow up may appear with **(M)+(P)** !

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ + \text{ various "good" boundary conditions} \end{cases}$$

- Theorem: [D. Schmitt-MP, 90'] One can find 'polynomial' nonlinearities f, g satisfying **(P)** and

$$\textbf{(M)}\ f + g \leq 0, \ \textit{and also}: \exists \lambda \in [0, 1[, f + \lambda g \leq 0,$$

for which $T^* < +\infty$ with

$$\lim_{t \to T^*} \|u_1(t)\|_{L^\infty(\Omega)} = \lim_{t \to T^*} \|u_2(t)\|_{L^\infty(\Omega)} = +\infty.$$

- Blow up may appear even in space dimension $N = 1$ (with high degree polynomial nonlinearities)
- Blow up may appear with any superquadratic growth $2 + \epsilon$ for the f_i (with high dimension N). Optimal ! [D. Schmitt-MP, 22']

To proceed:

Look for *weak solutions* which are allowed to go out of $L^\infty(\Omega)$ from time to time or even often ("Incomplete blow up").

We ask the nonlinearities to be at least in $L^1(Q_T)$:

$$f_i(u) \in L^1(Q_T)$$

and the solution is understood in the sense of distributions or of the integral formula :

$$u_i(t) = S_{d_i}(t)u_i^0 + \int_0^t S_{d_i}(t-s)f_i(u(s))ds,$$

where $S_{d_i}(t)$ is the semigroup generated by the Neumann Laplacian $-d_i\Delta$.

An L^1-approach

$$
(S)\begin{cases}
\forall i = 1, ..., m \\
\partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) \\
\partial_\nu u_i = 0 \\
u_i(0,\cdot) = u_i^0(\cdot) \geq 0.
\end{cases}
$$

- L^1-**Theorem.** [MP 03'] Assume **(P)+ (M')** hold. Assume moreover that the following a priori estimate holds:

$$
\forall i = 1, ..., m, \int_{Q_T} |f_i(u)| \leq C(T) < +\infty, \ \forall T \in (0, +\infty).
$$

Then, there exists a global weak solution for System (S), even for all $u_0 \in L^1(\Omega)^{+m}$!

An L^1-approach

$$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) \\ \partial_\nu u_i = 0 \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

- L^1-**Theorem.** [MP 03'] Assume **(P)+ (M')** hold. Assume moreover that the following a priori estimate holds:

  $$\forall i = 1, ..., m, \int_{Q_T} |f_i(u)| \leq C(T) < +\infty, \ \forall T \in (0, +\infty).$$

  Then, there exists a global weak solution for System (S), even for all $u_0 \in L^1(\Omega)^{+m}$!
- Proof involves L^1-properties of the heat operator and truncations techniques : for $T_k(r) := \inf\{r, k\}$, we use the equations satisfied by $T_k(u_i + \eta \sum_{j \neq i} u_j), \eta$ small.

Main ingredients in the proof of the L^1-theorem

- Approximating f_i by $f_i^n := \frac{f_i}{1+(\sum_j |f_j|)/n}$ and u_i^0 by $u_i^{0n} := \inf\{(u_i^0), n\}$
  $\mapsto$ global approximate solutions u_i^n with $\|f_i^n(u^n)\|_{L^1(Q_T)}$ bounded independently of n

$$(S)\begin{cases} \partial_t u_i^n - d_i \Delta u_i^n = f_i^n(u_1^n, ..., u_m^n) \ on\ (0,\infty)\times\Omega, \\ \partial_\nu u_i^n = 0 \ on\ (0,\infty)\times\partial\Omega, \\ u_i^n(0,\cdot) = u_i^{0n} \geq 0, \end{cases}$$

Main ingredients in the proof of the L^1-theorem

- Approximating f_i by $f_i^n := \frac{f_i}{1+(\sum_j |f_j|)/n}$ and u_i^0 by $u_i^{0n} := \inf\{(u_i^0), n\}$
  $\mapsto$ global approximate solutions u_i^n with $\|f_i^n(u^n)\|_{L^1(Q_T)}$ bounded independently of n

$$(S)\begin{cases} \partial_t u_i^n - d_i \Delta u_i^n = f_i^n(u_1^n, ..., u_m^n) \text{ on } (0,\infty)\times\Omega, \\ \partial_\nu u_i^n = 0 \text{ on } (0,\infty)\times\partial\Omega, \\ u_i^n(0,\cdot) = u_i^{0n} \geq 0, \end{cases}$$

- Compactness of the mapping
  $(g, w_0) \in L^1(Q_T)\times L^1(\Omega) \mapsto w \in L^1(Q_T)$ where

$$\partial_t w - d\Delta w = g \text{ on } Q_T,\ w(0,\cdot) = w_0,\ \partial_\nu w = 0 \text{ on } \partial\Omega.$$

so that $u_i^n \to u_i$ in $L^1(Q_T)$ and a.e. as $n \to +\infty$

Main ingredients in the proof of the L^1-theorem

- Approximating f_i by $f_i^n := \frac{f_i}{1+(\sum_j |f_j|)/n}$ and u_i^0 by $u_i^{0n} := \inf\{(u_i^0), n\}$
  $\mapsto$ global approximate solutions u_i^n with $\|f_i^n(u^n)\|_{L^1(Q_T)}$ bounded independently of n

$$(S)\left\{\begin{array}{l} \partial_t u_i^n - d_i \Delta u_i^n = f_i^n(u_1^n, ..., u_m^n) \; on \; (0,\infty)\times\Omega, \\ \partial_\nu u_i^n = 0 \; on \; (0,\infty)\times\partial\Omega, \\ u_i^n(0,\cdot) = u_i^{0n} \geq 0, \end{array}\right.$$

- Compactness of the mapping
  $(g, w_0) \in L^1(Q_T)\times L^1(\Omega) \mapsto w \in L^1(Q_T)$ where

$$\partial_t w - d\Delta w = g \; on \; Q_T, \; w(0,\cdot) = w_0, \; \partial_\nu w = 0 \; on \; \partial\Omega.$$

  so that $u_i^n \to u_i$ in $L^1(Q_T)$ and a.e. as $n \to +\infty$

- Proof involves L^1-type estimates of the heat operator like

$$\int_{[0\leq u_i^n \leq k]} d_i |\nabla u_i^n|^2 \leq k \left[\int_{Q_T} |f_i^n(u^n)| + \int_\Omega u_i^{0n}\right].$$

L^1-Theorem applies to many situations

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 e^{u_2^2} \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 e^{u_2^2} \end{cases}$$

- Easy $L^1(Q_T)$-estimate of the nonlinearity :

$$\int_\Omega u_1(T) + \int_{Q_T} u_1 e^{u_2^2} = \int_\Omega u_1^0.$$

L^1-Theorem applies to many situations

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 e^{u_2^2} \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 e^{u_2^2} \end{cases}$$

- Easy $L^1(Q_T)$-estimate of the nonlinearity :

$$\int_\Omega u_1(T) + \int_{Q_T} u_1 e^{u_2^2} = \int_\Omega u_1^0.$$

- $\Rightarrow$ Global existence of weak solutions

L^1-Theorem applies to many situations

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 e^{u_2^2} \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 e^{u_2^2} \end{cases}$$

- Easy $L^1(Q_T)$-estimate of the nonlinearity :

$$\int_\Omega u_1(T) + \int_{Q_T} u_1 e^{u_2^2} = \int_\Omega u_1^0.$$

- $\Rightarrow$ Global existence of weak solutions
- Existence holds for any $u_1^0, u_2^0 \in L^1(\Omega)^+$!!

L^1-Theorem applies to many situations

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 e^{u_2^2} \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 e^{u_2^2} \end{cases}$$

- Easy $L^1(Q_T)$-estimate of the nonlinearity :

$$\int_\Omega u_1(T) + \int_{Q_T} u_1 e^{u_2^2} = \int_\Omega u_1^0.$$

- $\Rightarrow$ Global existence of weak solutions
- Existence holds for any $u_1^0, u_2^0 \in L^1(\Omega)^+$!!
- Recall that the equation

$$\partial_t u_2 - \Delta u_2 = e^{u_2^2},\ u_2(0) = u_2^0,$$

does not have even local solutions in general when $u_2^0 \in L^1(\Omega)^+$ only.

L^1-Theorem applies to many situations

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1 e^{u_2^2} \\ \partial_t u_2 - d_2 \Delta u_2 = u_1 e^{u_2^2} \end{cases}$$

- Easy $L^1(Q_T)$-estimate of the nonlinearity :

$$\int_\Omega u_1(T) + \int_{Q_T} u_1 e^{u_2^2} = \int_\Omega u_1^0.$$

- $\Rightarrow$ Global existence of weak solutions
- Existence holds for any $u_1^0, u_2^0 \in L^1(\Omega)^+$!!
- Recall that the equation

$$\partial_t u_2 - \Delta u_2 = e^{u_2^2},\ u_2(0) = u_2^0,$$

  does not have even local solutions in general when $u_2^0 \in L^1(\Omega)^+$ only.
- OPEN PROBLEM: are the solutions classical ?

L^1-approach applies to many situations

- Like to the example of finite-time blow up in $L^\infty(\Omega)$:

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ +bdy \text{ and initial conditions and} \end{cases}$$

(M) $f_1 + f_2 \leq 0,$ *and also* $: \ \exists \, \lambda \neq 1, f_1 + \lambda f_2 \leq 0,$

L^1-approach applies to many situations

- Like to the example of finite-time blow up in $L^\infty(\Omega)$:

$$\left\{\begin{array}{l} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ + \textit{bdy and initial conditions and} \end{array}\right.$$

(M) $f_1 + f_2 \leq 0$, *and also* : $\exists \lambda \neq 1, f_1 + \lambda f_2 \leq 0$,

- $\partial_t(u_1 + u_2) - \Delta(d_1 u_1 + d_2 u_2) - (f_1(u) + f_2(u)) = 0.$

L^1-approach applies to many situations

- Like to the example of finite-time blow up in $L^\infty(\Omega)$:

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ +\textit{bdy and initial conditions and} \end{cases}$$

(M) $f_1 + f_2 \leq 0$, *and also* : $\exists\, \lambda \neq 1, f_1 + \lambda f_2 \leq 0,$

- $\partial_t(u_1 + u_2) - \Delta(d_1 u_1 + d_2 u_2) - (f_1(u) + f_2(u)) = 0.$
- With $\partial_\nu u_1 = 0 = \partial_\nu u_2$ on $\partial\Omega$, we deduce : $\int_\Omega \partial_t(u_1 + u_2) \ + \ 0 \ + \ \int_\Omega -(f_1(u) + f_2(u)) = 0.$

L^1-approach applies to many situations

- Like to the example of finite-time blow up in $L^\infty(\Omega)$:

$$\left\{\begin{array}{l} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ +bdy \textit{ and initial conditions and} \end{array}\right.$$

(M) $f_1 + f_2 \leq 0$, *and also* : $\exists \lambda \neq 1, f_1 + \lambda f_2 \leq 0,$

- $\partial_t(u_1 + u_2) - \Delta(d_1 u_1 + d_2 u_2) - (f_1(u) + f_2(u)) = 0.$
- With $\partial_\nu u_1 = 0 = \partial_\nu u_2$ on $\partial\Omega$, we deduce :
  $\int_\Omega \partial_t(u_1 + u_2) \; + \; 0 \; + \; \int_\Omega -(f_1(u) + f_2(u)) = 0.$
- $\Rightarrow \int_\Omega (u_1 + u_2)(T) + \int_{Q_T} |f_1(u) + f_2(u)| = \int_\Omega (u_1^0 + u_2^0).$

L^1-approach applies to many situations

- Like to the example of finite-time blow up in $L^\infty(\Omega)$:

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ +\textit{bdy and initial conditions and} \end{cases}$$

(M) $f_1 + f_2 \leq 0$, *and also* : $\exists \lambda \neq 1, f_1 + \lambda f_2 \leq 0$,

- $\partial_t(u_1 + u_2) - \Delta(d_1 u_1 + d_2 u_2) - (f_1(u) + f_2(u)) = 0$.
- With $\partial_\nu u_1 = 0 = \partial_\nu u_2$ on $\partial\Omega$, we deduce : $\int_\Omega \partial_t(u_1 + u_2) \ + \ 0 \ + \ \int_\Omega -(f_1(u) + f_2(u)) = 0$.
- $\Rightarrow \ \int_\Omega (u_1 + u_2)(T) + \int_{Q_T} |f_1(u) + f_2(u)| = \int_\Omega (u_1^0 + u_2^0)$.
- $\int_{Q_T} |f_1(u) + f_2(u)| \leq C \, [= C(\|u_1^0\|_{L^1}, \|u_2^0\|_{L^1})]$.

L^1-approach applies to many situations

- Like to the example of finite-time blow up in $L^\infty(\Omega)$:

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ +\textit{bdy and initial conditions and} \end{cases}$$

(M) $f_1 + f_2 \leq 0$, *and also* : $\exists \lambda \neq 1, f_1 + \lambda f_2 \leq 0,$

- $\partial_t(u_1 + u_2) - \Delta(d_1 u_1 + d_2 u_2) - (f_1(u) + f_2(u)) = 0.$
- With $\partial_\nu u_1 = 0 = \partial_\nu u_2$ on $\partial\Omega$, we deduce : $\int_\Omega \partial_t(u_1 + u_2) \; + \; 0 \; + \; \int_\Omega -(f_1(u) + f_2(u)) = 0.$
- $\Rightarrow \quad \int_\Omega (u_1 + u_2)(T) + \int_{Q_T} |f_1(u) + f_2(u)| = \int_\Omega (u_1^0 + u_2^0).$
- $\int_{Q_T} |f_1(u) + f_2(u)| \leq C \; [= C(\|u_1^0\|_{L^1}, \|u_2^0\|_{L^1})].$
- Similarly $\int_{Q_T} |f_1(u) + \lambda f_2(u)| \leq C, \; \lambda \neq 1.$

L^1-approach applies to many situations

- Like to the example of finite-time blow up in $L^\infty(\Omega)$:

$$\begin{cases} \partial_t u_1 - d_1 \Delta u_1 = f_1(u_1, u_2) \\ \partial_t u_2 - d_2 \Delta u_2 = f_2(u_1, u_2) \\ +\mathit{bdy\ and\ initial\ conditions\ and} \end{cases}$$

(M) $f_1 + f_2 \leq 0,$ *and also* $: \ \exists \lambda \neq 1, f_1 + \lambda f_2 \leq 0,$

- $\partial_t(u_1 + u_2) - \Delta(d_1 u_1 + d_2 u_2) - (f_1(u) + f_2(u)) = 0.$
- With $\partial_\nu u_1 = 0 = \partial_\nu u_2$ on $\partial\Omega$, we deduce:
  $\int_\Omega \partial_t(u_1 + u_2) \ + \ 0 \ + \ \int_\Omega -(f_1(u) + f_2(u)) = 0.$
- $\Rightarrow \ \int_\Omega (u_1 + u_2)(T) + \int_{Q_T} |f_1(u) + f_2(u)| = \int_\Omega (u_1^0 + u_2^0).$
- $\int_{Q_T} |f_1(u) + f_2(u)| \leq C \, [= C(\|u_1^0\|_{L^1}, \|u_2^0\|_{L^1})].$
- Similarly $\int_{Q_T} |f_1(u) + \lambda f_2(u)| \leq C, \ \lambda \neq 1.$
- $\Rightarrow \ \int_{Q_T} |f_1(u)|, \ \int_{Q_T} |f_2(u)| \leq C.$

L^1-Theorem applies to many situations

More generally, the same method applies if there exists an invertible matrix Q with nonnegative entries such that

$$\forall r \in [0,\infty)^m,\ Q\, f(r) \leq [1 + \sum_{1\leq i\leq m} r_i]\mathrm{b},$$

for some $\mathrm{b} \in \boldsymbol{R}^m, f = (f_1, ..., f_m)^t$.
In other words:
if there are m independent inequalities between the f_i's (not necessarily triangular)

Case of strictly less than m inequalities

$$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) \\ \partial_\nu u_i = 0 \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

- On the other hand, what about the system (S) with only [[**(P)** $+$ strictly less than m inequalities]], ...i.e. without a priori $L^1(Q_T)$-estimates on $f_i(u)$???

Case of strictly less than m inequalities

$$(S) \begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) \\ \partial_\nu u_i = 0 \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

- On the other hand, what about the system (S) with only [[**(P)** + strictly less than m inequalities]], ...i.e. without a priori $L^1(Q_T)$-estimates on $f_i(u)$???
- This is the case for the evolution of the concentrations of m chemical species $U_i, i = 1, ..., m$ undergoing reversible reaction together with diffusion, namely with $p_i, q_i \in \{0\} \cup [1, +\infty)$:

$$p_1 U_1 + p_2 U_2 + ... + p_m U_m \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_1 U_1 + q_2 U_2 + ... + q_m U_m,$$

Case of strictly less than m inequalities

$$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) \\ \partial_\nu u_i = 0 \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. \end{cases}$$

- On the other hand, what about the system (S) with only [[**(P)** + strictly less than m inequalities]], ...i.e. without a priori $L^1(Q_T)$-estimates on $f_i(u)$???
- This is the case for the evolution of the concentrations of m chemical species $U_i, i = 1, ..., m$ undergoing reversible reaction together with diffusion, namely with $p_i, q_i \in \{0\} \cup [1, +\infty)$:

$$p_1 U_1 + p_2 U_2 + ... + p_m U_m \ \underset{k^-}{\overset{k^+}{\rightleftharpoons}} \ q_1 U_1 + q_2 U_2 + ... + q_m U_m,$$

- The concentrations $u_i(t, x)$ of U_i satisfy a system of type (S) when the state laws are given by
  - the mass action kinetics for the reaction,
  - the (linear) Fick's law for the diffusion.

Evolution models for reversible chemistry

- Principle of mass action kinetics: the rate of a reaction is proportional to the concentration of the reactants [P.Waage, C.M.Guldberg,1864].

Evolution models for reversible chemistry

- Principle of mass action kinetics: the rate of a reaction is proportional to the concentration of the reactants [P.Waage, C.M.Guldberg,1864].
- $U_1 \ \rightarrow \ U_2$ with $u_i(t) =$ concentration of U_i

$$u_1'(t) = -u_2'(t) = -k\, u_1(t) \text{ for some } k > 0.$$

Evolution models for reversible chemistry

- Principle of mass action kinetics: the rate of a reaction is proportional to the concentration of the reactants [P.Waage, C.M.Guldberg,1864].
- $U_1 \;\rightarrow\; U_2$ with $u_i(t) =$ concentration of U_i

$$u_1'(t) = -u_2'(t) = -k\, u_1(t) \text{ for some } k > 0.$$

- $U_1 + p_2 U_2 \;\rightarrow\; q_3 U_3$

$$u_1'(t) = p_2^{-1} u_2'(t) = -q_3^{-1} u_3'(t) = -k u_1\, u_2^{p_2}.$$

Evolution models for reversible chemistry

- Principle of mass action kinetics: the rate of a reaction is proportional to the concentration of the reactants [P.Waage, C.M.Guldberg,1864].
- $U_1 \;\to\; U_2$ with $u_i(t) =$ concentration of U_i

$$u_1'(t) = -u_2'(t) = -k\, u_1(t) \text{ for some } k > 0.$$

- $U_1 + p_2 U_2 \;\to\; q_3 U_3$

$$u_1'(t) = p_2^{-1} u_2'(t) = -q_3^{-1} u_3'(t) = -k u_1 u_2^{p_2}.$$

- $U_1 + p_2 U_2 \;\underset{k^-}{\overset{k^+}{\rightleftharpoons}}\; q_3 U_3$

$$u_1' = p_2^{-1} u_2' = -q_3^{-1} u_3' = -k^+ u_1 u_2^{p_2} + k^- u_3^{q_3}.$$

Evolution models for reversible chemistry

- Principle of mass action kinetics: the rate of a reaction is proportional to the concentration of the reactants [P.Waage, C.M.Guldberg,1864].
- $U_1 \;\to\; U_2$ with $u_i(t) =$ concentration of U_i
$$u_1'(t) = -u_2'(t) = -k\, u_1(t) \text{ for some } k > 0.$$
- $U_1 + p_2 U_2 \;\to\; q_3 U_3$
$$u_1'(t) = p_2^{-1} u_2'(t) = -q_3^{-1} u_3'(t) = -k u_1 u_2^{p_2}.$$
- $U_1 + p_2 U_2 \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_3 U_3$
$$u_1' = p_2^{-1} u_2' = -q_3^{-1} u_3' = -k^+ u_1 u_2^{p_2} + k^- u_3^{q_3}.$$
- $U_1 + p_2 U_2 \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_2 U_2 + q_3 U_3$
$$u_1' = (p_2 - q_2)^{-1} u_2' = -q_3^{-1} u_3' = -k^+ u_1 u_2^{p_2} + k^- u_3^{q_3}.$$

Evolution models for reversible chemistry

- $p_1 U_1 + p_2 U_2 + ... + p_m U_m \overset{k^+}{\underset{k^-}{\rightleftharpoons}} q_1 U_1 + q_2 U_2 + ... + q_m U_m.$

$$u_i' = (q_i - p_i)[k^+ \prod_j u_j^{p_j} - k^- \prod_j u_j^{q_j}], \quad i = 1, ..., m.$$

Evolution models for reversible chemistry

- $p_1 U_1 + p_2 U_2 + ... + p_m U_m \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_1 U_1 + q_2 U_2 + ... + q_m U_m.$

$$u_i' = (q_i - p_i)[k^+ \prod_j u_j^{p_j} - k^- \prod_j u_j^{q_j}], \quad i = 1, ..., m.$$

- When $u_i = u_i(t, x)$, $u_i'(t)$ is to be replaced by $\partial_t u_i + \nabla \cdot (u_i V_i)$ where V_i =velocity of the U_i-particules.

Evolution models for reversible chemistry

- $p_1 U_1 + p_2 U_2 + \ldots + p_m U_m \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_1 U_1 + q_2 U_2 + \ldots + q_m U_m.$

$$u_i' = (q_i - p_i)[k^+ \prod_j u_j^{p_j} - k^- \prod_j u_j^{q_j}], \quad i = 1, \ldots, m.$$

- When $u_i = u_i(t, x)$, $u_i'(t)$ is to be replaced by $\partial_t u_i + \nabla \cdot (u_i V_i)$ where V_i =velocity of the U_i-particules.
- Fick's law for the diffusion says : $u_i V_i = -d_i \nabla u_i, \quad d_i \in (0, +\infty)$.

Evolution models for reversible chemistry

- $p_1 U_1 + p_2 U_2 + \ldots + p_m U_m \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_1 U_1 + q_2 U_2 + \ldots + q_m U_m.$

$$u_i' = (q_i - p_i)[k^+ \prod_j u_j^{p_j} - k^- \prod_j u_j^{q_j}], \quad i = 1, \ldots, m.$$

- When $u_i = u_i(t, x)$, $u_i'(t)$ is to be replaced by $\partial_t u_i + \nabla \cdot (u_i V_i)$ where V_i =velocity of the U_i-particules.
- Fick's law for the diffusion says : $u_i V_i = -d_i \nabla u_i, \quad d_i \in (0, +\infty)$.
- Whence the global system for $u = (u_i)_{1 \le i \le m}$:

$$\begin{cases} \forall i = 1, \ldots, m, \\ \partial_t u_i - d_i \Delta u_i = (q_i - p_i) h(u), \\ h(u) = k^+ \prod_j u_j^{p_j} - k^- \prod_j u_j^{q_j}. \end{cases}$$

Evolution models for reversible chemistry

- $p_1 U_1 + p_2 U_2 + \ldots + p_m U_m \underset{k^-}{\overset{k^+}{\rightleftharpoons}} q_1 U_1 + q_2 U_2 + \ldots + q_m U_m.$

$$u_i' = (q_i - p_i)[k^+ \prod_j u_j^{p_j} - k^- \prod_j u_j^{q_j}], \quad i = 1, \ldots, m.$$

- When $u_i = u_i(t, x)$, $u_i'(t)$ is to be replaced by $\partial_t u_i + \nabla \cdot (u_i V_i)$ where V_i =velocity of the U_i-particules.
- Fick's law for the diffusion says : $u_i V_i = -d_i \nabla u_i, \quad d_i \in (0, +\infty)$.
- Whence the global system for $u = (u_i)_{1 \le i \le m}$:

$$\begin{cases} \forall i = 1, \ldots, m, \\ \partial_t u_i - d_i \Delta u_i = (q_i - p_i) h(u), \\ h(u) = k^+ \prod_j u_j^{p_j} - k^- \prod_j u_j^{q_j}. \end{cases}$$

- And we may add: $\partial_\nu u_i = 0$ on $\partial\Omega$ for all i.

Evolution model for the reversible chemistry

$$(S)\begin{cases} \forall\, i = 1, ..., m, \\ \partial_t u_i - d_i \Delta u_i = f_i(u) := (q_i - p_i) h(u), \\ h(u) := k^+ \Pi_j u_j^{p_j} - k^- \Pi_j u_j^{q_j}, \\ \partial_\nu u_i = 0,\ u_i(0, \cdot) = u_i^0. \end{cases}$$

- The nonlinearity $f = (f_i)$ is quasipositive.

Evolution model for the reversible chemistry

$$(S)\begin{cases} \forall\, i = 1, ..., m, \\ \partial_t u_i - d_i \Delta u_i = f_i(u) := (q_i - p_i) h(u), \\ h(u) := k^+ \Pi_j u_j^{p_j} - k^- \Pi_j u_j^{q_j}, \\ \partial_\nu u_i = 0,\ u_i(0,\cdot) = u_i^0. \end{cases}$$

- The nonlinearity $f = (f_i)$ is quasipositive.
- There are (only) $m - 1$ independent (in)equalities:

$$(q_j - p_j) f_i + (p_i - q_i) f_j = 0, \quad i \in I, \quad j \in I$$

$$I := \{i = 1, ..., m;\ q_i - p_i < 0\}, \quad J := \{j = 1, ..., m;\ q_j - p_j > 0\}.$$

Evolution model for the reversible chemistry

$$
(S)\begin{cases}
\forall\, i = 1, ..., m, \\
\partial_t u_i - d_i \Delta u_i = f_i(u) := (q_i - p_i) h(u), \\
h(u) := k^+ \Pi_j u_j^{p_j} - k^- \Pi_j u_j^{q_j}, \\
\partial_\nu u_i = 0,\ u_i(0,\cdot) = u_i^0.
\end{cases}
$$

- The nonlinearity $f = (f_i)$ is quasipositive.
- There are (only) $m - 1$ independent (in)equalities:

$$(q_j - p_j) f_i + (p_i - q_i) f_j = 0, \quad i \in I, \quad j \in I$$

$$I := \{i = 1, ..., m;\ q_i - p_i < 0\}, \quad J := \{j = 1, ..., m;\ q_j - p_j > 0\}.$$

- There is an entropy inequality: if $k^+ = 1 = k^-$

$$
\begin{cases}
\sum_i (\log u_i) f_i(u) & = h(u) \sum_i (\log u_i)(q_i - p_i) \\
 & = h(u)[\log \Pi_i u_i^{q_i} - \log \Pi_i u_i^{p_i}] \leq 0,
\end{cases}
$$

$$\Rightarrow \quad \partial_t \int_\Omega u_i (\log u_i - 1) \leq 0 \ \text{ (entropy decrease)}.$$

Existence of global renormalized solutions

$$(S)\begin{cases} \forall\, i = 1, ..., m, \\ \partial_t u_i - d_i \Delta u_i = f_i(u) := (q_i - p_i) h(u), \\ h(u) := k^+ \Pi_j u_j^{p_j} - k^- \Pi_j u_j^{q_j}, \\ \partial_\nu u_i = 0,\ u_i(0,\cdot) = u_i^0 \log u_i^0 \in L^1(\Omega). \end{cases}$$

- Let u^ϵ be the solution of $\partial_t u_i^\epsilon - d_i \Delta u_i^\epsilon = f_i(u^\epsilon)/[1 + \epsilon \sum_j |f_j(u^\epsilon)|]$.
- **THEOREM** [J. Fischer, '2014] The approximate solution u^ϵ converges (up to a subsequence) on Q_∞ to some u with

$$u_i \in L^\infty(0,T;L^1(\Omega)), \sqrt{u_i} \in L^2(0,T;H^1(\Omega)),\ \forall T > 0,$$

such that for all $\xi : [0,\infty)^m \to \boldsymbol{R}$ compactly supported

$$\partial_t \xi(u) = \sum_i \partial_i \xi(u) \partial_t u_i = \sum_i \partial_i \xi(u)[d_i \Delta u_i + f_i(u)],$$

in a weak sense against test-functions.

Existence of global renormalized solutions

$$(S)\begin{cases} \forall\, i = 1, ..., m, \\ \partial_t u_i - d_i \Delta u_i = f_i(u) := (q_i - p_i) h(u), \\ h(u) := k^+ \Pi_j u_j^{p_j} - k^- \Pi_j u_j^{q_j}, \\ \partial_\nu u_i = 0,\; u_i(0,\cdot) = u_i^0 \log u_i^0 \in L^1(\Omega). \end{cases}$$

- Let u^ϵ be the solution of $\partial_t u_i^\epsilon - d_i \Delta u_i^\epsilon = f_i(u^\epsilon)/[1 + \epsilon \sum_j |f_j(u^\epsilon)|]$.
- **THEOREM** [J. Fischer, '2014] The approximate solution u^ϵ converges (up to a subsequence) on Q_∞ to some u with

$$u_i \in L^\infty(0,T; L^1(\Omega)), \sqrt{u_i} \in L^2(0,T; H^1(\Omega)),\ \forall T > 0,$$

  such that for all $\xi : [0,\infty)^m \to \boldsymbol{R}$ compactly supported

$$\partial_t \xi(u) = \sum_i \partial_i \xi(u) \partial_t u_i = \sum_i \partial_i \xi(u)[d_i \Delta u_i + f_i(u)],$$

  in a weak sense against test-functions.
- Solutions "à la Di Perna-Lions". Note $\partial_i \xi(u) f_i(u) \in L^\infty$!

Existence of global renormalized solutions

$$(S)\begin{cases} \forall\, i = 1, \dots, m, \\ \partial_t u_i - d_i \Delta u_i = f_i(u) := (q_i - p_i) h(u), \\ h(u) := k^+ \Pi_j u_j^{p_j} - k^- \Pi_j u_j^{q_j}, \\ \partial_\nu u_i = 0,\ u_i(0,\cdot) = u_i^0 \log u_i^0 \in L^1(\Omega). \end{cases}$$

- Let u^ϵ be the solution of $\partial_t u_i^\epsilon - d_i \Delta u_i^\epsilon = f_i(u^\epsilon)/[1 + \epsilon \sum_j |f_j(u^\epsilon)|]$.
- **THEOREM** [J. Fischer, '2014] The approximate solution u^ϵ converges (up to a subsequence) on Q_∞ to some u with

  $$u_i \in L^\infty(0,T; L^1(\Omega)), \sqrt{u_i} \in L^2(0,T; H^1(\Omega)),\ \forall T > 0,$$

  such that for all $\xi : [0,\infty)^m \to \boldsymbol{R}$ compactly supported

  $$\partial_t \xi(u) = \sum_i \partial_i \xi(u) \partial_t u_i = \sum_i \partial_i \xi(u)[d_i \Delta u_i + f_i(u)],$$

  in a weak sense against test-functions.
- Solutions "à la Di Perna-Lions". Note $\partial_i \xi(u) f_i(u) \in L^\infty$!
- Strong use of the entropy dissipation in the proof.

Existence of global renormalized solutions

$$(S)\begin{cases} \forall\, i = 1, ..., m, \\ \partial_t u_i - d_i \Delta u_i = f_i(u) := (q_i - p_i) h(u), \\ h(u) := k^+ \Pi_j u_j^{p_j} - k^- \Pi_j u_j^{q_j}, \\ \partial_\nu u_i = 0,\ u_i(0, \cdot) = u_i^0 \log u_i^0 \in L^1(\Omega). \end{cases}$$

- Let u^ϵ be the solution of $\partial_t u_i^\epsilon - d_i \Delta u_i^\epsilon = f_i(u^\epsilon)/[1 + \epsilon \sum_j |f_j(u^\epsilon)|]$.
- **THEOREM** [J. Fischer, '2014] The approximate solution u^ϵ converges (up to a subsequence) on Q_∞ to some u with

$$u_i \in L^\infty(0, T; L^1(\Omega)), \sqrt{u_i} \in L^2(0, T; H^1(\Omega)),\ \forall T > 0,$$

such that for all $\xi : [0, \infty)^m \to \boldsymbol{R}$ compactly supported

$$\partial_t \xi(u) = \sum_i \partial_i \xi(u) \partial_t u_i = \sum_i \partial_i \xi(u)[d_i \Delta u_i + f_i(u)],$$

in a weak sense against test-functions.
- Solutions "à la Di Perna-Lions". Note $\partial_i \xi(u) f_i(u) \in L^\infty$!
- Strong use of the entropy dissipation in the proof.
- OPEN PROBLEM: What about classical solutions ???

Existence of global renormalized solutions

$$
(S)\begin{cases}
\forall\, i = 1, ..., m, \\
\partial_t u_i - d_i \Delta u_i = f_i(u) := (q_i - p_i) h(u), \\
h(u) := k^+ \Pi_j u_j^{p_j} - k^- \Pi_j u_j^{q_j}, \\
\partial_\nu u_i = 0,\ u_i(0,\cdot) = u_i^0 \log u_i^0 \in L^1(\Omega).
\end{cases}
$$

- Let u^ϵ be the solution of $\partial_t u_i^\epsilon - d_i \Delta u_i^\epsilon = f_i(u^\epsilon)/[1 + \epsilon \sum_j |f_j(u^\epsilon)|]$.
- **THEOREM** [J. Fischer, '2014] The approximate solution u^ϵ converges (up to a subsequence) on Q_∞ to some u with

$$u_i \in L^\infty(0, T; L^1(\Omega)), \sqrt{u_i} \in L^2(0, T; H^1(\Omega)),\ \forall T > 0,$$

  such that for all $\xi : [0, \infty)^m \to \boldsymbol{R}$ compactly supported

$$\partial_t \xi(u) = \sum_i \partial_i \xi(u) \partial_t u_i = \sum_i \partial_i \xi(u)[d_i \Delta u_i + f_i(u)],$$

  in a weak sense against test-functions.
- Solutions "à la Di Perna-Lions". Note $\partial_i \xi(u) f_i(u) \in L^\infty$!
- Strong use of the entropy dissipation in the proof.
- OPEN PROBLEM: What about classical solutions ???
- Even : what about weak solutions ?? $\Leftrightarrow$ $f_i(u) \in L^1(Q_T)$???

A surprising L^2-estimate for the systems **(P)+(M')**

$$
(S)\begin{cases}
\forall i = 1, ..., m & \\
\partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\
\partial_\nu u_i = 0 & \text{on } \Sigma_T \\
u_i(0,\cdot) = u_i^0(\cdot) \geq 0. &
\end{cases}
$$

- L^2-**Theorem.** Assume **(P)+(M')**. Then, the following a priori estimate holds for the solutions of (S):

$$
\forall T > 0, \ \int_{Q_T} \sum_{i=1}^m u_i^2 \leq C \int_\Omega \left(\sum_{i=1}^m u_i^0 \right)^2, \ C = C(T, (d_i)).
$$

A surprising L^2-estimate for the systems **(P)+(M')**

$$(S) \begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$

- L^2-**Theorem.** Assume **(P)+(M')**. Then, the following a priori estimate holds for the solutions of (S):

$$\forall T > 0, \; \int_{Q_T} \sum_{i=1}^m u_i^2 \leq C \int_\Omega \left(\sum_{i=1}^m u_i^0 \right)^2, \; C = C(T, (d_i)).$$

- The proof uses only the sum of the equations

$$\partial_t(\sum_i u_i) - \Delta(\sum_i d_i u_i) \leq 0.$$

Idea of the proof of the L^2-estimate

- $$\partial_t \left(\sum_i u_i \right) - \Delta \left(\sum_i d_i u_i \right) \leq 0.$$

Idea of the proof of the L^2-estimate

- $$\partial_t \left(\sum_i u_i \right) - \Delta \left(\sum_i d_i u_i \right) \leq 0.$$

- $$\partial_t W - \Delta \left(a\, W \right) \leq 0, \quad W = \sum_i u_i, \quad a = \frac{\sum_i d_i u_i}{\sum_i u_i}$$

Idea of the proof of the L^2-estimate

- $$\partial_t \left(\sum_i u_i \right) - \Delta \left(\sum_i d_i u_i \right) \le 0.$$
- $$\partial_t W - \Delta (a\, W) \le 0, \quad W = \sum_i u_i, \quad a = \frac{\sum_i d_i u_i}{\sum_i u_i}$$
- $$0 \le \underline{d} = \min_i d_i \le a = \frac{\sum_i d_i u_i}{\sum_i u_i} \le \max_i d_i = \overline{d} < +\infty$$

Idea of the proof of the L^2-estimate

- $$\partial_t \left(\sum_i u_i \right) - \Delta \left(\sum_i d_i u_i \right) \leq 0.$$

- $$\partial_t W - \Delta \left(a\, W \right) \leq 0, \quad W = \sum_i u_i, \;\; a = \frac{\sum_i d_i u_i}{\sum_i u_i}$$

- $$0 \leq \underline{d} = \min_i d_i \leq a = \frac{\sum_i d_i u_i}{\sum_i u_i} \leq \max_i d_i = \overline{d} < +\infty$$

- The operator $W \to \partial_t W - \Delta(aW)$ is not of divergence form and a is not continuous, but bounded from above and from below so that the operator is parabolic and this implies

$$\|W\|_{L^2(Q_T)} \leq C \|W_0\|_{L^2(\Omega)}.$$

Idea of the proof of the L^2-estimate

- $$\partial_t \left(\sum_i u_i\right) - \Delta \left(\sum_i d_i u_i\right) \leq 0.$$

- $$\partial_t W - \Delta (a W) \leq 0, \quad W = \sum_i u_i, \quad a = \frac{\sum_i d_i u_i}{\sum_i u_i}$$

- $$0 \leq \underline{d} = \min_i d_i \leq a = \frac{\sum_i d_i u_i}{\sum_i u_i} \leq \max_i d_i = \overline{d} < +\infty$$

- The operator $W \to \partial_t W - \Delta(aW)$ is not of divergence form and a is not continuous, but bounded from above and from below so that the operator is parabolic and this implies
$$\|W\|_{L^2(Q_T)} \leq C\|W_0\|_{L^2(\Omega)}.$$

- Seen on the dual operator $\psi \to -(\partial_t \psi + a\Delta\psi)$ which satisfies L^2-maximal regularity in terms of $\underline{d}, \overline{d}$.

A proof of the L^2-estimate by duality

- We multiply the inequality $\partial_t W - \Delta(a\,W) \leq 0$, by the solution $\psi \geq 0$ of the dual problem

$$\begin{cases} -(\partial_t \psi + a\Delta\psi) = \Theta \in C_0^\infty(Q_T)^+, \\ \psi(T) = 0,\ \partial_\nu \psi = 0. \end{cases}$$

A proof of the L^2-estimate by duality

- We multiply the inequality $\partial_t W - \Delta(a\, W) \leq 0$, by the solution $\psi \geq 0$ of the dual problem

$$\begin{cases} -(\partial_t \psi + a\Delta\psi) = \Theta \in C_0^\infty(Q_T)^+, \\ \psi(T) = 0,\ \partial_\nu \psi = 0. \end{cases}$$

- $\Rightarrow \int_{Q_T} W\Theta = \int_\Omega W_0 \psi(0) \leq \|W_0\|_{L^2(\Omega)} \|\psi(0)\|_{L^2(\Omega)}$.

A proof of the L^2-estimate by duality

- We multiply the inequality $\partial_t W - \Delta(a\,W) \leq 0$, by the solution $\psi \geq 0$ of the dual problem

$$\left\{ \begin{array}{l} -(\partial_t \psi + a\Delta\psi) = \Theta \in C_0^\infty(Q_T)^+, \\ \psi(T) = 0,\ \partial_\nu \psi = 0. \end{array} \right.$$

- $\Rightarrow \int_{Q_T} W\Theta = \int_\Omega W_0 \psi(0) \leq \|W_0\|_{L^2(\Omega)} \|\psi(0)\|_{L^2(\Omega)}$.
- And $\|\psi(0)\|_{L^2(\Omega)} \leq C(\underline{d}, \overline{d}, T)\|\Theta\|_{L^2(Q_T)}$, whence the L^2-estimate on W by duality.

A proof of the L^2-estimate by duality

- We multiply the inequality $\partial_t W - \Delta(a\,W) \le 0$, by the solution $\psi \ge 0$ of the dual problem

$$\begin{cases} -(\partial_t \psi + a\Delta\psi) = \Theta \in C_0^\infty(Q_T)^+, \\ \psi(T) = 0,\ \partial_\nu \psi = 0. \end{cases}$$

- $\Rightarrow \int_{Q_T} W\Theta = \int_\Omega W_0 \psi(0) \le \|W_0\|_{L^2(\Omega)} \|\psi(0)\|_{L^2(\Omega)}$.
- And $\|\psi(0)\|_{L^2(\Omega)} \le C(\underline{d}, \overline{d}, T)\|\Theta\|_{L^2(Q_T)}$, whence the L^2-estimate on W by duality.
- Indeed multiplying the equation in ψ by $-\Delta\psi$ gives $\int_{Q_T} (\Delta\psi)\partial_t \psi + a(\Delta\psi)^2 = -\int_{Q_T} \Delta\psi\,\Theta$.

A proof of the L^2-estimate by duality

- We multiply the inequality $\partial_t W - \Delta(a\, W) \leq 0$, by the solution $\psi \geq 0$ of the dual problem

$$\begin{cases} -(\partial_t \psi + a\Delta\psi) = \Theta \in C_0^\infty(Q_T)^+, \\ \psi(T) = 0, \ \partial_\nu \psi = 0. \end{cases}$$

- $\Rightarrow \int_{Q_T} W\Theta = \int_\Omega W_0 \psi(0) \leq \|W_0\|_{L^2(\Omega)} \|\psi(0)\|_{L^2(\Omega)}$.
- And $\|\psi(0)\|_{L^2(\Omega)} \leq C(\underline{d}, \overline{d}, T)\|\Theta\|_{L^2(Q_T)}$, whence the L^2-estimate on W by duality.
- Indeed multiplying the equation in ψ by $-\Delta\psi$ gives $\int_{Q_T} (\Delta\psi)\partial_t \psi + a(\Delta\psi)^2 = -\int_{Q_T} \Delta\psi\, \Theta$.
- $\int_{Q_T} (\Delta\psi)\partial_t\psi = -\int_{Q_T} \nabla\psi \partial_t \nabla\psi = -\frac{1}{2}\int_{Q_T} \partial_t |\nabla\psi|^2 = \frac{1}{2}\int_\Omega |\nabla\psi(0)|^2 \geq 0$.

A proof of the L^2-estimate by duality

- We multiply the inequality $\partial_t W - \Delta(a\, W) \leq 0$, by the solution $\psi \geq 0$ of the dual problem
$$\begin{cases} -(\partial_t \psi + a\Delta\psi) = \Theta \in C_0^\infty(Q_T)^+, \\ \psi(T) = 0,\ \partial_\nu \psi = 0. \end{cases}$$
- $\Rightarrow \int_{Q_T} W\Theta = \int_\Omega W_0 \psi(0) \leq \|W_0\|_{L^2(\Omega)} \|\psi(0)\|_{L^2(\Omega)}$.
- And $\|\psi(0)\|_{L^2(\Omega)} \leq C(\underline{d}, \overline{d}, T)\|\Theta\|_{L^2(Q_T)}$, whence the L^2-estimate on W by duality.
- Indeed multiplying the equation in ψ by $-\Delta\psi$ gives $\int_{Q_T} (\Delta\psi)\partial_t\psi + a(\Delta\psi)^2 = -\int_{Q_T} \Delta\psi\,\Theta$.
- $\int_{Q_T} (\Delta\psi)\partial_t\psi = -\int_{Q_T} \nabla\psi\partial_t\nabla\psi = -\frac{1}{2}\int_{Q_T} \partial_t|\nabla\psi|^2 = \frac{1}{2}\int_\Omega |\nabla\psi(0)|^2 \geq 0$.
- $\Rightarrow \underline{d}\|\Delta\psi\|^2_{L^2(Q_T)} \leq \|\Delta\psi\|_{L^2(Q_T)}\|\Theta\|_{L^2(Q_T)}$.

A proof of the L^2-estimate by duality

- We multiply the inequality $\partial_t W - \Delta(a\,W) \le 0$, by the solution $\psi \ge 0$ of the dual problem

$$\begin{cases} -(\partial_t \psi + a\Delta\psi) = \Theta \in C_0^\infty(Q_T)^+, \\ \psi(T) = 0,\ \partial_\nu \psi = 0. \end{cases}$$

- $\Rightarrow \int_{Q_T} W\Theta = \int_\Omega W_0\psi(0) \le \|W_0\|_{L^2(\Omega)} \|\psi(0)\|_{L^2(\Omega)}$.
- And $\|\psi(0)\|_{L^2(\Omega)} \le C(\underline{d}, \overline{d}, T)\|\Theta\|_{L^2(Q_T)}$, whence the L^2-estimate on W by duality.
- Indeed multiplying the equation in ψ by $-\Delta\psi$ gives $\int_{Q_T} (\Delta\psi)\partial_t\psi + a(\Delta\psi)^2 = -\int_{Q_T} \Delta\psi\,\Theta$.
- $\int_{Q_T} (\Delta\psi)\partial_t\psi = -\int_{Q_T} \nabla\psi\partial_t\nabla\psi = -\frac{1}{2}\int_{Q_T} \partial_t|\nabla\psi|^2 = \frac{1}{2}\int_\Omega |\nabla\psi(0)|^2 \ge 0$.
- $\Rightarrow \underline{d}\|\Delta\psi\|^2_{L^2(Q_T)} \le \|\Delta\psi\|_{L^2(Q_T)}\|\Theta\|_{L^2(Q_T)}$.
- This implies an $L^2(Q_T)$-estimate on $\Delta\psi$, then on $\partial_t\psi$ and then the $L^2(\Omega)$-estimate on $\psi(0)$.

Extensions of the L^2-estimate for such systems

- This L^2-estimate is robust !

Extensions of the L^2-estimate for such systems

- This L^2-estimate is robust !
- Variable coefficients $d_i = d_i(t, x)$, nonlinear diffusions $-\Delta d_i(u_i)$
- $W_0 \in L^1(\Omega)$ only !: The L^2-estimate may be localized for $\partial_t W - \Delta(aW) \leq 0$, $\underline{d} \leq a \leq \overline{d}$.

$$\|W\|_{L^2((\tau,T)\times\Omega)} \leq \frac{C(\underline{d}, \overline{d}, T)}{\tau^{N/4}} \|W_0\|_{L^1(\Omega)}.$$

- [J.A. Cañizo, L. Desvillettes, K. Fellner]: there exists $\epsilon(N) > 0$ such that

$$\|W\|_{L^{2+\epsilon}(Q_T)} \leq C\|W_0\|_{L^{2+\epsilon}(\Omega)}.$$

Applications to quadratic systems

$$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0, \, u^0 \in L^2(\Omega)^m. \end{cases}$$

- **Corollary of the L^1 and L^2 Theorems.** Assume **(P)+(M')** and the f_i are at most quadratic, i.e.

$$\forall 1 \leq i \leq m, \ \ \forall r \in [0,\infty)^2, \ \ |f_i(r)| \leq C[1 + \sum r_j^2].$$

Then, (S) has a global weak solution.

Applications to quadratic systems

$$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_T \\ \partial_\nu u_i = 0 & \text{on } \Sigma_T \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0,\ u^0 \in L^2(\Omega)^m. \end{cases}$$

- **Corollary of the L^1 and L^2 Theorems.** Assume **(P)+(M')** and the f_i are at most quadratic, i.e.

  $$\forall 1 \leq i \leq m,\ \ \forall r \in [0,\infty)^2,\ \ |f_i(r)| \leq C[1 + \sum r_j^2].$$

  Then, (S) has a global weak solution.

- **Proof .** By L^2-estimate + at most quadratic growth of f_i, we have the a priori estimate for the solutions of (S)

  $$\forall 1 \leq i \leq m,\ \ \int_{Q_T} |f_i(u)| \leq C(T),$$

  Then, we apply the L^1-theorem.

Applications to quadratic systems

$$(S)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \textit{in } Q_T \\ \partial_\nu u_i = 0 & \textit{on } \Sigma_T \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0, u^0 \in L^2(\Omega)^m. \end{cases}$$

- **Corollary of the L^1 and L^2 Theorems.** Assume **(P)+(M')** and the f_i are at most quadratic, i.e.

$$\forall 1 \leq i \leq m, \ \ \forall r \in [0, \infty)^2, \ \ |f_i(r)| \leq C[1 + \sum r_j^2].$$

  Then, (S) has a global weak solution.

- **Proof .** By L^2-estimate + at most quadratic growth of f_i, we have the a priori estimate for the solutions of (S)

$$\forall 1 \leq i \leq m, \ \ \int_{Q_T} |f_i(u)| \leq C(T),$$

  Then, we apply the L^1-theorem.

- But quite more has recently be proved !

A recent main result for systems with quadratic growth

$$(S_\infty)\begin{cases} \forall i = 1, ..., m \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \textit{in } Q_\infty, \\ \partial_\nu u_i = 0 & \textit{on } \Sigma_\infty, \\ u_i(0,\cdot) = u_i^0(\cdot) \in L^\infty(\Omega)^+. \end{cases}$$

- Assume f satisfies **(P)+(M')** and $\forall\, i = 1, ..., m$

$$|f_i(r)| \leq K[1 + |\sum_i r_i|^{2+\epsilon}],\ \forall\, r = (r_1, ..., r_m) \in [0, +\infty)^m.$$

A recent main result for systems with quadratic growth

$$
(S_\infty)\begin{cases}
\forall i = 1, ..., m & \\
\partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_\infty, \\
\partial_\nu u_i = 0 & \text{on } \Sigma_\infty, \\
u_i(0,\cdot) = u_i^0(\cdot) \in L^\infty(\Omega)^+. &
\end{cases}
$$

- Assume f satisfies **(P)+(M')** and $\forall\, i = 1, ..., m$

$$
|f_i(r)| \le K[1 + |\sum_i r_i|^{2+\epsilon}],\ \forall\, r = (r_1, ..., r_m) \in [0, +\infty)^m.
$$

- Then if ϵ is small enough, there exists a global classical solution to S_∞.

A recent main result for systems with quadratic growth

$$(S_\infty)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i\, \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_\infty, \\ \partial_\nu u_i = 0 & \text{on } \Sigma_\infty, \\ u_i(0,\cdot) = u_i^0(\cdot) \in L^\infty(\Omega)^+. & \end{cases}$$

- Assume f satisfies **(P)+(M')** and $\forall\, i = 1, ..., m$

$$|f_i(r)| \leq K[1 + |\sum_i r_i|^{2+\epsilon}],\ \forall\, r = (r_1, ..., r_m) \in [0, +\infty)^m.$$

- Then if ϵ is small enough, there exists a global classical solution to S_∞.
- To be compared with the blow up examples...: ϵ being given, one can construct solutions which blow up in $L^\infty(\Omega)$ in finite time, but the dimension has to be chosen large enough. ($\epsilon = \epsilon(N)$ above).

A recent main result for systems with quadratic growth

$$(S_\infty)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & in\ Q_\infty, \\ \partial_\nu u_i = 0 & on\ \Sigma_\infty, \\ u_i(0,\cdot) = u_i^0(\cdot) \in L^\infty(\Omega)^+. & \end{cases}$$

- Assume f satisfies **(P)+(M')** and $\forall\, i = 1, ..., m$

$$|f_i(r)| \leq K[1 + |\sum_i r_i|^{2+\epsilon}],\ \forall\, r = (r_1, ..., r_m) \in [0, +\infty)^m.$$

- Then if ϵ is small enough, there exists a global classical solution to S_∞.
- To be compared with the blow up examples...: ϵ being given, one can construct solutions which blow up in $L^\infty(\Omega)$ in finite time, but the dimension has to be chosen large enough. ($\epsilon = \epsilon(N)$ above).
- Includes the famous Lotka-Volterra system in any dimension !.

A recent main result for systems with quadratic growth

$$(S_\infty)\begin{cases} \forall i = 1, \dots, m & \\ \partial_t u_i - d_i \,\Delta u_i = f_i(u_1, u_2, \dots, u_m) & \text{in } Q_\infty, \\ \partial_\nu u_i = 0 & \text{on } \Sigma_\infty, \\ u_i(0,\cdot) = u_i^0(\cdot) \in L^\infty(\Omega)^+. & \end{cases}$$

- Assume f satisfies **(P)+(M')** and $\forall\, i = 1, \dots, m$

$$|f_i(r)| \leq K[1 + |\sum_i r_i|^{2+\epsilon}],\ \forall\, r = (r_1, \dots, r_m) \in [0, +\infty)^m.$$

- Then if ϵ is small enough, there exists a global classical solution to S_∞.
- To be compared with the blow up examples...: ϵ being given, one can construct solutions which blow up in $L^\infty(\Omega)$ in finite time, but the dimension has to be chosen large enough. ($\epsilon = \epsilon(N)$ above).
- Includes the famous Lotka-Volterra system in any dimension !.
- Includes : $U_1 + U_3 \ \underset{k^-}{\overset{k^+}{\rightleftharpoons}}\ U_2 + U_4$

A recent main result for systems with quadratic growth

$$(S_\infty)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \, \Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_\infty, \\ \partial_\nu u_i = 0 & \text{on } \Sigma_\infty, \\ u_i(0, \cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$

- The above quadratic result is due to K.Fellner, J. Morgan and B. Q. Tang (in 2 papers 2020 and 2021).

A recent main result for systems with quadratic growth

$$(S_\infty)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i \Delta u_i = f_i(u_1, u_2, ..., u_m) & \textit{in } Q_\infty, \\ \partial_\nu u_i = 0 & \textit{on } \Sigma_\infty, \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$

- The above quadratic result is due to K.Fellner, J. Morgan and B. Q. Tang (in 2 papers 2020 and 2021).
- It relies in particular on a technique developed by Ya.I. Kanel (1990) where such a global existence result was proved in $\boldsymbol{R}^N$ and for equality in **(M)**.

A recent main result for systems with quadratic growth

$$(S_\infty)\begin{cases} \forall i = 1, ..., m & \\ \partial_t u_i - d_i\,\Delta u_i = f_i(u_1, u_2, ..., u_m) & \text{in } Q_\infty, \\ \partial_\nu u_i = 0 & \text{on } \Sigma_\infty, \\ u_i(0,\cdot) = u_i^0(\cdot) \geq 0. & \end{cases}$$

- The above quadratic result is due to K.Fellner, J. Morgan and B. Q. Tang (in 2 papers 2020 and 2021).
- It relies in particular on a technique developed by Ya.I. Kanel (1990) where such a global existence result was proved in $\boldsymbol{R}^N$ and for equality in **(M)**.
- And similar global existence results were also obtained in 2018 by Ph. Souplet and in 2019 by M.C. Caputo, Th. Goudon and A. Vasseur assuming an entropy dissipation (as for reversible chemistry)).

Ingredients of the proof of the quadratic theorem

- A surprising interpolation lemma whose idea is taken from Ya. Kanel in $\boldsymbol{R}^N$ and adapted by K. Fellner, J. Morgan and B.Q. Tang to the case of a bounded regular domain with homogeneous Neumann boundary conditions (it would also work with Dirichlet boundary conditions)

Ingredients of the proof of the quadratic theorem

- A surprising interpolation lemma whose idea is taken from Ya. Kanel in $\boldsymbol{R}^N$ and adapted by K. Fellner, J. Morgan and B.Q. Tang to the case of a bounded regular domain with homogeneous Neumann boundary conditions (it would also work with Dirichlet boundary conditions)
- Let w be the solution of the heat equation

$$\begin{cases} \partial_t w - d\Delta w = \Theta \textit{ in } Q_T, \\ \partial_\nu w = 0 \textit{ on } \Sigma_T, \quad w(0) = w_0, \\ \Theta \in L^\infty(Q_T),\ d \in (0, +\infty). \end{cases}$$

Ingredients of the proof of the quadratic theorem

- A surprising interpolation lemma whose idea is taken from Ya. Kanel in $\boldsymbol{R}^N$ and adapted by K. Fellner, J. Morgan and B.Q. Tang to the case of a bounded regular domain with homogeneous Neumann boundary conditions (it would also work with Dirichlet boundary conditions)
- Let w be the solution of the heat equation

$$\begin{cases} \partial_t w - d\Delta w = \Theta \text{ } in \text{ } Q_T, \\ \partial_\nu w = 0 \text{ } on \text{ } \Sigma_T, \quad w(0) = w_0, \\ \Theta \in L^\infty(Q_T), \text{ } d \in (0, +\infty). \end{cases}$$

- Assume that for some $H \in (0, +\infty), \gamma \in [0, 1)$

$$|w(t,x) - w(t,y)| \leq H|x-y|^\gamma, \quad \forall t \in (0,T), \text{ } x, y \in \Omega.$$

Ingredients of the proof of the quadratic theorem

- A surprising interpolation lemma whose idea is taken from Ya. Kanel in $\boldsymbol{R}^N$ and adapted by K. Fellner, J. Morgan and B.Q. Tang to the case of a bounded regular domain with homogeneous Neumann boundary conditions (it would also work with Dirichlet boundary conditions)
- Let w be the solution of the heat equation

$$\begin{cases} \partial_t w - d\Delta w = \Theta \text{ } in \text{ } Q_T, \\ \partial_\nu w = 0 \text{ } on \text{ } \Sigma_T, \quad w(0) = w_0, \\ \Theta \in L^\infty(Q_T), \text{ } d \in (0, +\infty). \end{cases}$$

- Assume that for some $H \in (0, +\infty), \gamma \in [0, 1)$

$$|w(t,x) - w(t,y)| \leq H|x-y|^\gamma, \quad \forall t \in (0,T), \text{ } x, y \in \Omega.$$

- Then there exists $B, C \in (0, +\infty)$ such that, for all $t \in [0, T]$

$$(\gamma\text{-Lemma}) \text{ } \|\nabla w(t)\|_{L^\infty(\Omega)^N} \leq C\|\nabla w_0\|_{L^\infty(\Omega)^N} + BH^{\frac{1}{2-\gamma}}\|\Theta\|_{L^\infty(Q_T)}^{\frac{1-\gamma}{2-\gamma}}.$$

Ingredients of the proof of the quadratic theorem

- A surprising interpolation lemma whose idea is taken from Ya. Kanel in $\boldsymbol{R}^N$ and adapted by K. Fellner, J. Morgan and B.Q. Tang to the case of a bounded regular domain with homogeneous Neumann boundary conditions (it would also work with Dirichlet boundary conditions)
- Let w be the solution of the heat equation

$$\begin{cases} \partial_t w - d\Delta w = \Theta \text{ in } Q_T, \\ \partial_\nu w = 0 \text{ on } \Sigma_T, \quad w(0) = w_0, \\ \Theta \in L^\infty(Q_T),\ d \in (0, +\infty). \end{cases}$$

- Assume that for some $H \in (0, +\infty), \gamma \in [0, 1)$

$$|w(t,x) - w(t,y)| \leq H|x-y|^\gamma, \quad \forall t \in (0, T),\ x, y \in \Omega.$$

- Then there exists $B, C \in (0, +\infty)$ such that, for all $t \in [0, T]$

$$(\gamma\text{-Lemma})\ \|\nabla w(t)\|_{L^\infty(\Omega)^N} \leq C\|\nabla w_0\|_{L^\infty(\Omega)^N} + BH^{\frac{1}{2-\gamma}}\|\Theta\|_{L^\infty(Q_T)}^{\frac{1-\gamma}{2-\gamma}}.$$

- $\gamma = 0:\ \|\nabla w(t)\|_{L^\infty(\Omega)^N} \leq C\|\nabla w_0\|_{L^\infty(\Omega)^N} + BH^{\frac{1}{2}}\|\Theta\|_{L^\infty}^{\frac{1}{2}}, H = 2\|w\|_{L^\infty}$

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

- We apply this γ-lemma to $V := \int_0^t \sum_i d_i u_i$

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

- We apply this γ-lemma to $V := \int_0^t \sum_i d_i u_i$
- Rewrite $\partial_t \sum_i u_i - \Delta(\sum_i d_i u_i) = 0$ in several ways:

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

- We apply this γ-lemma to $V := \int_0^t \sum_i d_i u_i$
- Rewrite $\partial_t \sum_i u_i - \Delta(\sum_i d_i u_i) = 0$ in several ways:
- $$(\mathrm{V}_1) \qquad \sum_i u_i(t) - \Delta V = \sum_i u_i^0.$$

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

- We apply this γ-lemma to $V := \int_0^t \sum_i d_i u_i$
- Rewrite $\partial_t \sum_i u_i - \Delta(\sum_i d_i u_i) = 0$ in several ways:
- $$(\mathrm{V}_1) \quad \sum_i u_i(t) - \Delta V = \sum_i u_i^0.$$
- $$(\mathrm{V}_2) \quad \partial_t V - \Delta V = \sum_i (d_i - 1) u_i + \sum_i u_i^0.$$

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

- We apply this γ-lemma to $V := \int_0^t \sum_i d_i u_i$
- Rewrite $\partial_t \sum_i u_i - \Delta(\sum_i d_i u_i) = 0$ in several ways:
- $$(\mathrm{V}_1) \quad \sum_i u_i(t) - \Delta V = \sum_i u_i^0.$$
- $$(\mathrm{V}_2) \quad \partial_t V - \Delta V = \sum_i (d_i - 1) u_i + \sum_i u_i^0.$$
- $$(\mathrm{V}_3) \quad \frac{1}{a} \partial_t V - \Delta V = \sum_i u_i^0,$$

where $a = \frac{\sum_i d_i u_i}{\sum_i u_i}$, $\underline{d} \le a \le \overline{d}$.

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

$$
\begin{array}{ll}
(\mathrm{V}_1) & \sum_i u_i(t) - \Delta V = \sum_i u_i^0. \\
(\mathrm{V}_2) & \partial_t V - \Delta V = \sum_i (d_i - 1) u_i + \sum_i u_i^0. \\
(\mathrm{V}_3) & \frac{1}{a}\partial_t V - \Delta V = \sum_i u_i^0.
\end{array}
$$

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

$$
\begin{array}{ll}
(\mathrm{V}_1) & \sum_i u_i(t) - \Delta V = \sum_i u_i^0. \\
(\mathrm{V}_2) & \partial_t V - \Delta V = \sum_i (d_i - 1) u_i + \sum_i u_i^0. \\
(\mathrm{V}_3) & \frac{1}{a}\partial_t V - \Delta V = \sum_i u_i^0.
\end{array}
$$

- By (V_1), if we bound $\|\Delta V\|_{L^\infty(Q_{T^*})} \Rightarrow$ global existence !

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

$$
\begin{array}{ll}
(\mathrm{V}_1) & \sum_i u_i(t) - \Delta V = \sum_i u_i^0. \\
(\mathrm{V}_2) & \partial_t V - \Delta V = \sum_i (d_i - 1) u_i + \sum_i u_i^0. \\
(\mathrm{V}_3) & \frac{1}{a}\partial_t V - \Delta V = \sum_i u_i^0.
\end{array}
$$

- By (V_1), if we bound $\|\Delta V\|_{L^\infty(Q_{T^*})} \Rightarrow$ global existence !
- By applying the Krylov-Safanov estimates to (V_3), we obtain a uniform Hölder-estimate for V on Q_{T^*} (depending on $\underline{d}, \overline{d}$).

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

$$
\begin{array}{ll}
(V_1) & \sum_i u_i(t) - \Delta V = \sum_i u_i^0. \\
(V_2) & \partial_t V - \Delta V = \sum_i (d_i - 1) u_i + \sum_i u_i^0. \\
(V_3) & \frac{1}{a}\partial_t V - \Delta V = \sum_i u_i^0.
\end{array}
$$

- By (V_1), if we bound $\|\Delta V\|_{L^\infty(Q_{T^*})}$ $\Rightarrow$ global existence !
- By applying the Krylov-Safanov estimates to (V_3), we obtain a uniform Hölder-estimate for V on Q_{T^*} (depending on $\underline{d}, \overline{d}$).
- Apply the γ-lemma to V in (V_2) !!!!

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

$$
\begin{array}{ll}
(\mathrm{V}_1) & \sum_i u_i(t) - \Delta V = \sum_i u_i^0. \\
(\mathrm{V}_2) & \partial_t V - \Delta V = \sum_i (d_i - 1) u_i + \sum_i u_i^0. \\
(\mathrm{V}_3) & \frac{1}{a}\partial_t V - \Delta V = \sum_i u_i^0.
\end{array}
$$

- By (V$_1$), if we bound $\|\Delta V\|_{L^\infty(Q_{T^*})}$ $\Rightarrow$ global existence !
- By applying the Krylov-Safanov estimates to (V$_3$), we obtain a uniform Hölder-estimate for V on Q_{T^*} (depending on $\underline{d}, \overline{d}$).
- Apply the γ-lemma to V in (V$_2$) !!!!
- $+$ some other more classical interpolation lemma for (V$_2$) and for the equations in u_i

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

$$
\begin{array}{ll}
(\mathrm{V}_1) & \sum_i u_i(t) - \Delta V = \sum_i u_i^0. \\
(\mathrm{V}_2) & \partial_t V - \Delta V = \sum_i (d_i - 1) u_i + \sum_i u_i^0. \\
(\mathrm{V}_3) & \frac{1}{a} \partial_t V - \Delta V = \sum_i u_i^0.
\end{array}
$$

- By (V_1), if we bound $\|\Delta V\|_{L^\infty(Q_{T^*})} \Rightarrow$ global existence !
- By applying the Krylov-Safanov estimates to (V_3), we obtain a uniform Hölder-estimate for V on Q_{T^*} (depending on $\underline{d}, \overline{d}$).
- Apply the γ-lemma to V in (V_2) !!!!
- $+$ some other more classical interpolation lemma for (V_2) and for the equations in u_i
- $+$ use the assumption of quadratic growth of the f_i.

Ideas of the proof of the quadratic theorem: case $\sum_i f_i(u) = 0$

$$
\begin{array}{ll}
(\mathrm{V}_1) & \sum_i u_i(t) - \Delta V = \sum_i u_i^0. \\
(\mathrm{V}_2) & \partial_t V - \Delta V = \sum_i (d_i - 1) u_i + \sum_i u_i^0. \\
(\mathrm{V}_3) & \frac{1}{a}\partial_t V - \Delta V = \sum_i u_i^0.
\end{array}
$$

- By (V_1), if we bound $\|\Delta V\|_{L^\infty(Q_{T^*})}$ $\Rightarrow$ global existence !
- By applying the Krylov-Safanov estimates to (V_3), we obtain a uniform Hölder-estimate for V on Q_{T^*} (depending on $\underline{d}, \overline{d}$).
- Apply the γ-lemma to V in (V_2) !!!!
- $+$ some other more classical interpolation lemma for (V_2) and for the equations in u_i
- $+$ use the assumption of quadratic growth of the f_i.
- and it works...

Application to the Lotka-Volterra system

- Applies to the (quadratic) Lotka-Volterra system: for all $u^0 \in L^\infty(\Omega)^{+m}$, there exists a global classical solution to the system: For all $i = 1, ..., m$,

$$\partial_t u_i - d_i \Delta u_i = e_i u_i + \left(\sum_{1 \le j \le m} p_{ij} u_j \right) u_i =: f_i(u),$$

where $e_i \in \boldsymbol{R}$, $p_{ij} \in \boldsymbol{R}$ + "Dissipation".

Application to the Lotka-Volterra system

- Applies to the (quadratic) Lotka-Volterra system: for all $u^0 \in L^\infty(\Omega)^{+m}$, there exists a global classical solution to the system: For all $i = 1, ..., m$,

$$\partial_t u_i - d_i \Delta u_i = e_i u_i + \left(\sum_{1 \leq j \leq m} p_{ij} u_j \right) u_i =: f_i(u),$$

where $e_i \in \boldsymbol{R}$, $p_{ij} \in \boldsymbol{R}$ + "Dissipation".

- "Dissipation": we assume that, for some $(a_i) \in (0, +\infty)^m$

$$\forall \xi \in [0, \infty)^m, \quad \sum_{i,j=1}^{m} a_i p_{ij} \xi_i \xi_j \leq 0$$

$$\Rightarrow \sum_i a_i f_i(u) \leq \sum_i a_i e_i u_i \ \text{(whence } \textbf{(M')}\text{)}$$