Dissipation in quantum mechanics : the Schrödinger-Langevin equation

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Table of contents

In classical mechanics

Dissipation with $\mu > 0$ of an ODE : for all $t \geq 0$,

$$E(t) + \mu \int_0^t \underbrace{D(s)}_{\geq 0} \, ds = E(0).$$

In classical mechanics

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Example : the damped harmonic oscillator

$$\ddot{X} + \mu \dot{X} + \omega^2 X = 0,$$

with dissipated energy

$$\underbrace{\frac{1}{2}\dot{X}(t)^2}_{=E_{\text{kin}}(t)} + \underbrace{\frac{\omega^2}{2}X(t)^2}_{=E_{\text{pot}}(t)} + \mu \underbrace{\int_0^t \dot{X}(s)^2 ds}_{=2\int_0^t E_{\text{kin}}(s) ds} = E(0).$$

Damping in quantum mechanics

Consider (NLS) equation

$$i\partial_t\psi + \Delta\psi + |\psi|^2\psi = 0$$

with usual invariants for all $t \geq 0$,

$$M(t) = \|\psi(t,.)\|_{L^2(\mathbb{R}^d)} = M(0),$$

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi|^2 - \frac{1}{4}\int_{\mathbb{R}^d} |\psi|^4 = E(0).$$

Damping in quantum mechanics

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Typical damping in the mathematical literature (blow-up criterion Fibich 2001, control theory) :

$$i\partial_t \psi + \Delta\psi + |\psi|^2\psi + i\mu\psi = 0,$$

with $\mu > 0$. No mass-conservation property :

$$\|\psi(t,.)\|_{L^2(\mathbb{R}^d)} = e^{-\mu t}\|\psi_0\|_{L^2(\mathbb{R}^d)}.$$

Schrödinger-Langevin Equation

Kostin 1972 derived a nonlinear Schrödinger equation

$$i\partial_t\psi + \Delta\psi = \frac{\mu}{2i}\psi \log\left(\frac{\psi}{\psi^*}\right)$$

which formally preserves mass.

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which formally preserves mass.
Few mathematical investigation :

- ill-posedness when ψ tends to 0,
- multivaluation of the argument $\psi = |\psi|e^{i\theta}$ with

$$\theta = \frac{1}{2i}\log\left(\frac{\psi}{\psi^*}\right).$$

Lopez & Montejo-Gamez 2011 : local existence of strong solutions away from zero in a bouded domain.

Logarithmic Schrödinger equation

Białynicki-Birula & Mycielski 1976 introduced (logNLS) :

$$i\partial_t\psi + \Delta\psi = \lambda\psi \log|\psi|^2, \qquad \text{(logNLS)}$$

with $\lambda \in \mathbb{R}^*$.

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- Mass and energy

$$\begin{aligned} M(\psi) &= \|\psi\|_{L^2(\mathbb{R}^d)}. \\ H(\psi) &= \|\nabla\psi\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |\psi|^2(\log|\psi|^2 - 1)\mathrm{d}x. \end{aligned}$$

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- Gaussian solutions :
  - $\underline{\lambda > 0 :}$ dispersion at rate $(t\sqrt{logt})^{-d/2}$,
  - $\underline{\lambda < 0 :}$ existence of stationary solutions called **Gaussons**.

Logarithmic Schrödinger equation

$$i\partial_t \psi + \Delta\psi = \lambda\psi \log|\psi|^2$$

- Focusing case ($\lambda > 0$)
  - Cauchy Problem : Cazenave & Haraux 1980, Cazenave 2003. Global existence in the energy space

$$W := \left\{\psi \in H^1(\mathbb{R}^d) \mid x \mapsto |\psi(x)|^2 \log|\psi(x)|^2 \in L^1(\mathbb{R}^d)\right\}.$$

  - Orbital stability of the Gausson (Cazenave 1983, Ardila 2016), existence of multi-Gaussons (Ferriere 2021).

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  - Orbital stability of the Gausson (Cazenave 1983, Ardila 2016), existence of multi-Gaussons (Ferriere 2021).
- Defocusing case ($\lambda > 0$)
  - Global existence in the weighted Sobolev space $H^1(\mathbb{R}^d) \cap \mathcal{F}(H^1)$ (Guerrero, López & Nieto 2010, Carles & Gallagher 2018).
  - Uniform behavior of solutions (Carles & Gallagher 2018) and universal dispersion at a scale $(t\sqrt{\log t})^{-d/2}$.

Schrödinger-Langevin equation

Nassar 1985 introduced (SL) :

$$\boxed{i\partial_t\psi + \Delta\psi = \lambda\psi\log|\psi|^2 + \frac{\mu}{2i}\psi\log\left(\frac{\psi}{\psi^*}\right)}$$

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- Quantum mechanics, cosmology, statistical mechanics : Zander & al. 2015, Mousavi & al. 2019, Chavanis 2017.

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- Quantum mechanics, cosmology, statistical mechanics : Zander & al. 2015, Mousavi & al. 2019, Chavanis 2017.
- Fluid formulation with Madelung transform $\psi = \sqrt{\rho}e^{iS}$ and $u = \nabla S$: isothermal Euler-Korteweg system with damping

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u\otimes u) + \lambda\nabla\rho + \mu\rho u = \dfrac{1}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \end{cases}$$

Schrödinger-Langevin equation

$$i\partial_t \psi + \Delta \psi = \lambda \psi \log |\psi|^2 + \frac{\mu}{2i} \psi \log \left(\frac{\psi}{\psi^*} \right) \tag{SL}$$

<u>Analysis on $\mathbb{R}^d$</u>

- Mass conservation $\|\rho(t,\cdot)\|_{L^1} = \|\rho_0\|_{L^1}$ and energy dissipation :

$$\underbrace{\frac{1}{2}\int_{\mathbb{R}^d} \rho |u|^2}_{=E_{\mathrm{kin}}(t)} + \underbrace{\int_{\mathbb{R}^d} \left(|\nabla \sqrt{\rho}|^2 + \lambda \rho (\log \rho - 1) \right)}_{=E_{\mathrm{pot}}(t)} + \mu \underbrace{\int_0^t \int_{\mathbb{R}^d} \rho |u|^2 ds}_{=2\int_0^t E_{\mathrm{kin}}(s) ds} \leq E_0 .$$

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- Gaussian particular solutions.

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- Gaussian particular solutions.
- Dynamics (C. 2021) :
  - $\lambda > 0$: Universal dispersion in $t^{-d/4}$, slowlier than the linear and logarithmic one.
  - $\lambda < 0$: Convergence towards the Gausson $e^{\lambda|x|^2}$.

Equation on $\mathbb{T}^d$

$$i\partial_t\psi + \Delta\psi = \lambda\psi\log|\psi|^2 + \frac{\mu}{2i}\psi\log\left(\frac{\psi}{\psi^*}\right) \tag{SL}$$

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- Global existence on H^1 (compactness) when $\mu = 0$.

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- Global existence on H^1 (compactness) when $\mu = 0$.
- No Gaussian calculus.
- For initial data made of a single Fourier mode $u(0,x) = \rho e^{im\cdot x}$ (with $\rho > 0$ and $m \in \mathbb{Z}^d$), (logNLS) has the unique plane-wave solution

$$\nu_m(t,x) = \rho e^{i(m\cdot x - \omega t)}$$

with $\omega = |m|^2 + 2\lambda\log\rho$, and the only stationary plane wave solutions of (SL) are the constant plane wave functions of the form

$$\nu = \rho e^{-2i\lambda\log\rho/\mu}, \quad \rho > 0,$$

which belongs to any Sobolev space on the torus $\Rightarrow$ stability ?

Fourier basis and Sobolev spaces

For $u \in L^2(\mathbb{T}^d)$ we associate the Fourier coefficients u_n for $n \in \mathbb{Z}^d$:

$$u_n = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(x) e^{-inx} \mathrm{d}x \quad \text{s.t.} \quad u = \sum_{n \in \mathbb{Z}^d} u_n e^{in\cdot x}.$$

The functions $(|u_n|)_{n \in \mathbb{Z}^d}$ are called the **actions**. Specific notation : $\langle u \rangle := u_0$. We define the Sobolev spaces $H^s(\mathbb{T}^d)$ with the norm

$$\|u\|_{H^s} = \left(\sum_{n \in \mathbb{Z}^d} \left(1 + |n|^2\right)^s |u_n|^2 \right)^{\frac{1}{2}}.$$

$H^s(\mathbb{T}^d)$ is an algebra when $s > d/2$: there exists a constant $C_s > 0$ such that for all $u,\ v \in H^s$, we have

$$\|uv\|_{H^s} \leq C_s \|u\|_{H^s} \|v\|_{H^s}. \tag{1}$$

Definition of the logarithm

By classical lifting theorem, we can define $a(t) > 0$ and $\theta(t) \in \mathbb{R}$ so that we have the following parametrization, valid for all u such that $\langle u \rangle \neq 0$:

$$u(t,x) = e^{i\theta(t)}(a(t) + w(t,x)), \quad \langle w \rangle = 0. \tag{2}$$

In this case, we can define the logarithm

$$\log(u(t,x)) := i\theta(t) + \log a(t) + \log\left(1 + \frac{w(t,x)}{a(t)}\right).$$

This application is well defined and smooth for curves on the domain

$$\mathcal{U}_s \;=\; \left\{ u = e^{i\theta}(a+w) \;\middle|\; (a,\theta,w), a > 0,\, \langle w \rangle = 0, \quad \left\|\frac{w}{a}\right\|_{H^s} < \frac{1}{C_s} \right\}.$$

Logarithmic Schrödinger equation

For $\mu = 0$:

$$i\partial_t \psi + \Delta\psi = \lambda\psi \log|\psi|^2. \tag{logNLS}$$

This equation is associated with the energy

$$H(\psi, \psi^*) := \|\nabla\psi\|_{L^2}^2 + \frac{\lambda}{(2\pi)^d} \int_{\mathbb{T}^d} |\psi(t,x)|^2 \left(\log|\psi(t,x)|^2 - 1\right) dx,$$

which is preserved for all time $t \geq 0$, as equation (logNLS) can be written

$$i\partial_t \psi = \frac{\partial H}{\partial \psi^*}(\psi, \psi^*).$$

The L^2 norm is also preserved along the dynamics, namely for all $t \geq 0$,

$$\|\psi(t,\cdot)\|_{L^2} = \|\psi^0\|_{L^2}.$$

Logarithmic Schrödinger equation

Theorem (C., Faou 2022)

Fix $m \in \mathbb{Z}^d$, $-\frac{1}{2} < \lambda_- < \lambda_+$, $\rho > 0$, $N > 1$. $\exists s_0 > 0$, $C \geq 1$, $\Lambda \subset [\lambda_-, \lambda_+]$ *of full measure such that* $\forall\, s \geq s_0$, $\forall \lambda \in \Lambda$, $\exists \varepsilon_0 > 0$ *such that if*

$$\|e^{-im.x}\psi^0 - \langle e^{-im.x}\psi^0\rangle\|_{H^s} = \varepsilon \leq \varepsilon_0, \quad \textit{and} \quad \|\psi^0\|_{L^2} = \rho,$$

then the solution ψ *of (logNLS) with* $\psi(0,x) = \psi^0(x) \in \mathcal{U}_s$ *satisfies*

$$\|e^{-im.x}\psi(t,\cdot\,) - \rho e^{i\theta(t)}\|_{H^s} \leq C\varepsilon \quad \textit{for} \quad t \leq \varepsilon^{-N},$$

and

$$|\dot{\theta} - (m^2 + 2\lambda \log \rho)| \leq C\varepsilon^2.$$

Sketch of the proof and limits

$\Rightarrow$ Stability analysis in H^s : strategy of Faou, Gauckler, Lubich (2013) for cubic NLS.

- Sketch of the proof :
  - use of invariants of (logNLS) in order to reduce the problem,
  - linearization of the logarithm around the plane wave,
  - use of standard Birkhoff normal theorems (Bambusi, Grébert 2006) to conclude.
- Limits :
  - result holds for *almost all* λ,
  - s_0 can be arbitrary large.

Numerics

We take

$$\psi_0(x) = \frac{1}{1 + 0.2\cos(x)}.$$

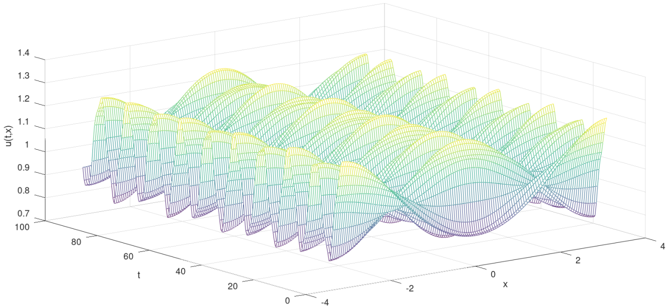

Figure – Solution of equation (logNLS) with initial datum ψ_0 ($\lambda = 0.5$).

Numerics

We recall that

$$\psi(t,x) = \sum_{n\in\mathbb{Z}} \psi_n(t)e^{inx}.$$

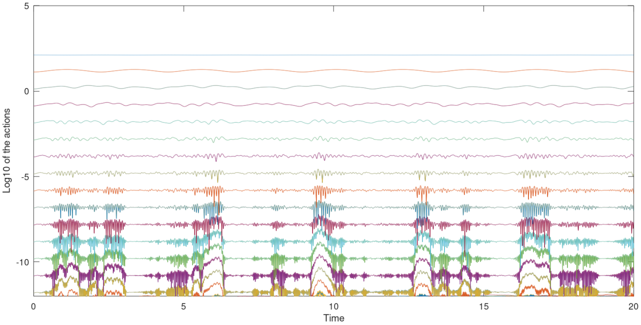

Figure – Evolution of the actions $|\psi_n|^2$ of the solution of equation (logNLS) with initial datum ψ_0 ($\lambda = 0.5$).

Schrödinger-Langevin equation

For $\mu > 0$, we now consider the Schrödinger-Langevin equation :

$$i\partial_t \psi + \Delta\psi = \lambda\psi \log|\psi|^2 + \frac{\mu}{2i}\psi \log\left(\frac{\psi}{\psi^*}\right). \tag{SL}$$

- Only stationary plane wave solutions are the constants :

$$\nu = \rho e^{-2i\lambda \log \rho/\mu}, \quad \rho > 0.$$

- Still preservation of the L^2 norm.
- Dissipation of energy $\Rightarrow$ no normal form result available.

Schrödinger-Langevin equation

Theorem (C., Faou 2022)

Let $s > \frac{d}{2}$, $\lambda > -\frac{1}{2}$, $\mu > 0$ *and* $\rho > 0$. *Then there exists* $\varepsilon_0 > 0$ *such that, if the initial datum satisfies* $\|\psi^0 - \langle\psi^0\rangle\|_{H^s} \leq \varepsilon_0$ *and* $\|\psi_0\|_{L^2} = \rho$ *then the solution of (SL) with* $\psi(0,.) = \psi^0 \in \mathcal{U}_s$ *satisfies for* $t \geq 0$,

$$\|\psi(t,.) - \rho e^{-2i\lambda \log \rho/\mu}\|_{H^s} \leq Ce^{-\alpha t}(1 + \beta t),$$

where

(i) *If* $\mu < 2\sqrt{1+2\lambda}$ *then* $\alpha = \frac{\mu}{2}$ *and* $\beta = 0$.

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(iii) *If* $\mu > 2\sqrt{1+2\lambda}$ *then* $\alpha = \frac{\mu}{2} - \left(\frac{\mu^2}{4} - 1 - 2\lambda\right)^{\frac{1}{2}} \in \left(0, \frac{\mu}{2}\right)$, *and if there exists* $n \geq 2$ *such that* $\mu^2 = 4n^2 + 8\lambda n$ *we have* $\beta = 1$, *and if this is not the case,* $\beta = 0$.

Sketch of the proof

- **Scaling factor** : if ψ is solution of (SL) with initial datum $\psi(0,x) = \psi^0$, then

$$\kappa\psi \exp\left(2i\frac{\lambda}{\mu}\log\kappa\left(1 - e^{-\mu t}\right)\right)$$

is also a solution of (SL) with initial datum $\psi(0,x) = \kappa\psi^0$. Thus it is sufficient to prove the result for $\rho = 1$.

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  is also a solution of (SL) with initial datum $\psi(0,x) = \kappa\psi^0$. Thus it is sufficient to prove the result for $\rho = 1$.

- **Elimination of the zero mode**

$$\psi = e^{i\theta}(a + w) \quad \text{with} \quad \langle w \rangle = 0.$$

  Preservation of the L^2 norm + Parseval :

$$a = \sqrt{1 - \|w\|_{L^2}^2}.$$

  Note that we can also control θ in terms of a and w $\Rightarrow$ $\psi \equiv w$.

Sketch of the proof

- **Analytic development of the logarithm** : On $\mathcal{U}_s$,

$$\log\left(1+\frac{w}{a}\right) = -\sum_{n\geq 1}\frac{1}{n}\left(-\frac{w}{a}\right)^n. \tag{3}$$

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- **Linearization** : we write w in Fourier basis, so

$$w = \sum_{j\neq 0} w_j e^{ij\cdot x}.$$

Projecting on the j-th mode, the equation of motion for w_j can be written

$$i\partial_t w_j = \left(|j|^2+\lambda+\frac{\mu}{2i}\right) w_j + \left(\lambda-\frac{\mu}{2i}\right)\overline{w}_{-j} + \frac{\partial\mathcal{P}}{\partial\overline{w}_j}(w,\overline{w}),$$

where

$$\text{with}\quad \mathcal{P}(w,\overline{w}) = \sum_{r\geq 3}\mathcal{P}_r(w,\overline{w})$$

and $\mathcal{P}_r(w,\overline{w})$ denotes a polynomial of degree r .

Sketch of the proof

Denoting $W_j = (w_j, \overline{w}_{-j})^T$ and $n = |j|^2 \geq 1$, we have

$$\dot{W}_j = -iA_n W_j + \mathcal{O}(\|W\|_{L^2}^2),$$

where

$$A_n = \begin{pmatrix} n + \lambda + \frac{\mu}{2i} & \lambda - \frac{\mu}{2i} \\ -\lambda - \frac{\mu}{2i} & -n - \lambda + \frac{\mu}{2i} \end{pmatrix}.$$

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In particular, the eigenvalues of the matrix $-iA_n$ are

$$-\frac{\mu}{2} \pm i\sqrt{n^2 + 2\lambda n - \frac{\mu^2}{4}}.$$

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In particular, the eigenvalues of the matrix $-iA_n$ are

$$-\frac{\mu}{2} \pm i\sqrt{n^2 + 2\lambda n - \frac{\mu^2}{4}}.$$

We can distinguish three cases :

- $\underline{4n^2 + 8\lambda n - \mu^2 > 0 :}$ in this case the eigenvalues are of the form

$$-\frac{\mu}{2} \pm i\delta_n, \quad \text{with} \quad \delta_n = \sqrt{n^2 + 2\lambda n - \frac{\mu^2}{4}} > 0.$$

Sketch of the proof

- $\underline{4n^2 + 8\lambda n - \mu^2 = 0 :}$ in this case $-\frac{\mu}{2}$ is a double eigenvalue. The matrix A_n can be put under the Jordan form

$$P_n^{-1} A_n P_n = \begin{pmatrix} \frac{\mu}{2i} & \lambda - \frac{\mu}{2i} \\ 0 & \frac{\mu}{2i} \end{pmatrix}.$$

- $\underline{4n^2 + 8\lambda n - \mu^2 < 0 :}$ note that for given λ and μ, this situation occurs only a finite number of times, as when n becomes large, $n^2 + 2\lambda n$ goes to $+\infty$. In this case the two eigenvalues are under for the form $-\alpha_n$ and $-\beta_n$, with

$$\alpha_n = \frac{\mu}{2} - \sqrt{\frac{\mu^2}{4} - n^2 - 2\lambda n}, \qquad \beta_n = \frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} - n^2 - 2\lambda n}.$$

Conclusion

- To conclude (for instance in the simplest case $\mu < 2\sqrt{1+2\lambda}$), denoting $V = P^{-1}W$ in the system

$$i\partial_t V = DV + R(V),$$

and $U = e^{\frac{\mu}{2}t}V$, we get that

$$i\partial_t U = \tilde{D}U + e^{\frac{\mu}{2}t}R(e^{-\frac{\mu}{2}t}V),$$

with $\tilde{D}$ real and R at least quadratic.

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$$i\partial_t U = \tilde{D}U + e^{\frac{\mu}{2}t}R(e^{-\frac{\mu}{2}t}V),$$

with $\tilde{D}$ real and R at least quadratic.

- In the end, comparison with the ODI

$$|\dot{y}| \leq Me^{-\frac{\alpha}{2}t}y^{3/2},$$

where $y(t) = \|U(t,.)\|_{H^s}$. Uniformly bounded under the condition $y(0) < (\alpha/M)^2$.

Remarks

- $\beta = 1$: Jordan block in the reduced dynamics.
- When λ is fixed, $\mu \to \infty$, the damping rate

$$\alpha = \frac{2\lambda + 1}{\mu} + \mathcal{O}\left(\frac{1}{\mu^3}\right)$$

  goes to 0. Thus a larger damping coefficient implies a slower relaxation to the equilibrium (overdamping).
- $\lambda > -\frac{1}{2}$ crucial. Numerical experiments for $\lambda = -\frac{1}{2}$ tend to show that the solution converges to a non trivial quasi-periodic solution.

Numerics

$$\psi_0(x) = \frac{1}{1 + 0.2\cos(x)}.$$

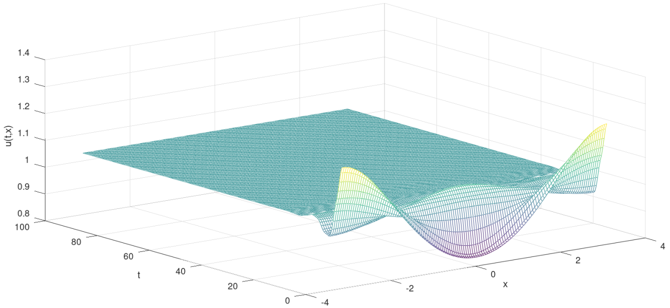

Figure – Solution of equation (SL) with initial datum ψ_0 ($\lambda = 0.5$, $\mu = 2$).

Numerics

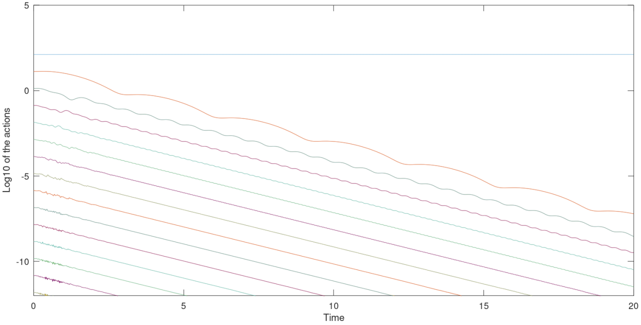

Figure – Evolution of the actions of solution of equation (SL) with initial datum ψ_0 ($\lambda = 0.5$, $\mu = 2$).

Numerics

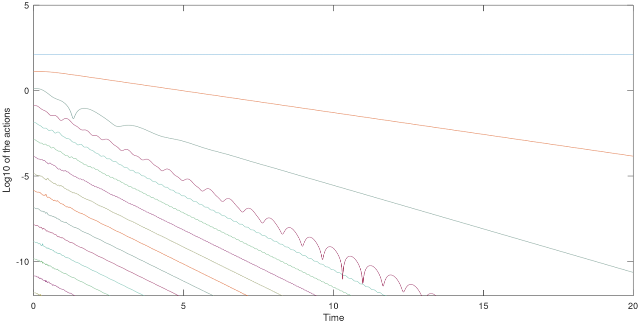

Figure – Evolution of the actions of solution of equation (SL) with initial datum ψ_0 ($\lambda = 0.5$, $\mu = 4$).

Numerics

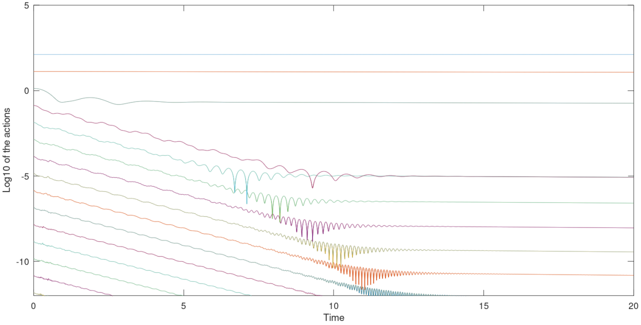

Figure – Evolution of the actions of solution of equation (SL) with initial datum ψ_0 ($\lambda = -0.5$, $\mu = 2$).

Perspectives

- Damped harmonic oscillator

$$i\partial_t\psi + \Delta\psi = |x|^2\psi + \frac{\mu}{2i}\psi \log\left(\frac{\psi}{\psi^*}\right).$$

- Other mass-preserving damping : linear damping

$$i\partial_t\psi + \Delta\psi = |x|^2\psi + \frac{\mu}{2i}\left(x\cdot\nabla\psi + \frac{d}{2}\psi\right),$$

  or Doebner-Goblin models

$$i\partial_t\psi + \Delta\psi = i\mu\left(\Delta\psi + \frac{|\nabla\psi|^2}{|\psi|^2}\psi\right).$$

- Damping for other dispersive PDEs (Klein-Gordon, KdV).

Thanks for your attention !

Sketch of the proof

- **Galilean invariance principle** : if ψ is a solution, then

$$\varphi(t,x) = \psi(t, x - vt)e^{-i\left(|v|^2 t/2 - v\cdot x\right)}$$

is also a solution for every $v \in \mathbb{Z}^d$. Hence

$$\nu_m(t, x + mt)e^{-i\left(|m|^2 t + m\cdot x\right)} = \rho e^{-2i\lambda t \log \rho} = \nu_0(t,x),$$

so we can restrict our attention to the case $m = 0$.

- **Scaling factor** : if ψ is a solution to (logNLS), then

$$\kappa\psi(t,x)e^{2it\lambda \log \kappa}, \quad \kappa > 0,$$

also solves (logNLS) with initial datum $\kappa\psi^0$, so we can take

$$\|\psi(t,\cdot)\|_{L^2} = \|\psi^0\|_{L^2} = 1,$$

which means that we can consider only the case $\rho = 1$.

Sketch of the proof

- **Elimination of the zero mode**

$$\psi = e^{i\theta}(a+w) \quad \text{with} \quad \langle w \rangle = 0.$$

Preservation of the L^2 norm

$$a = \sqrt{1 - \|w\|_{L^2}^2}.$$

Note that we can also control θ in terms of a and w :

$$\dot{\theta} + \mu\theta + 2\lambda \log(a) = \frac{1}{a}\mathrm{Re}\langle \mathcal{P}(w, w^*)\rangle, \tag{4}$$

- **Analytic development of the logarithm** : On $\mathcal{U}_s$,

$$\log\left(1 + \frac{w}{a}\right) = -\sum_{n\geq 1} \frac{1}{n}\left(-\frac{w}{a}\right)^n. \tag{5}$$

Sketch of the proof

The equation of motion for w can be written, for $j \in \mathbb{Z}^*$,

$$i\partial_t w_j = (|j|^2 + \lambda) w_j + \lambda \overline{w}_{-j} + \frac{\partial \mathcal{P}}{\partial \overline{w}_j}(w, \overline{w}),$$

where

$$\text{with} \quad \mathcal{P}(w, \overline{w}) = \sum_{r \geq 3} \mathcal{P}_r(w, \overline{w})$$

and $\mathcal{P}_r(w, \overline{w})$ denotes a polynomial of degree r of the form

$$\mathcal{P}_r(w, \overline{w}) = \sum_{p+q=r} \sum_{\substack{(j,\ell) \in \mathcal{Z}^p \times \mathcal{Z}^q \\ j_1 + \ldots + j_p - \ell_1 - \ldots - \ell_q = 0}} \mathcal{P}_{k,\ell} w_{j_1} \ldots w_{j_p} \overline{w}_{\ell_1} \ldots \overline{w}_{\ell_q}.$$

Sketch of the proof

- **Diagonalization of the linear part** Denoting $W_j = (w_j, \overline{w}_{-j})^T$,

$$\dot{W}_j = -iA_{|j|^2} W_j + \mathcal{O}(\|W\|^2_{L^2}),$$

where for $n = |j|^2 \geq 1$, we have

$$A_n = \begin{pmatrix} n+\lambda & \lambda \\ -\lambda & -n-\lambda \end{pmatrix}.$$

Lemma

Let $\lambda > -1/2$. Then, for all $n \geq 1$, the matrix A_n is diagonalized by a 2×2 matrix S_n that is real symplectic and hermitian and has condition number smaller than 2 :

$$S_n^{-1}(-iA_n)S_n = \begin{pmatrix} \Omega_n & 0 \\ 0 & -\Omega_n \end{pmatrix} \quad \textit{with} \quad \Omega_n = i\sqrt{n^2 + 2\lambda n}.$$

Conclusion

- **Non resonance condition and normal form** Let $r > 1$, $\exists \alpha = \alpha(r) > 0$ s.t. for $\lambda \in \Lambda$ there exists $\gamma > 0$, s.t. for all integers p, q with $p + q \leq r$ and for all $m = (m_1, \ldots, m_p) \in \mathbb{N}^p$ and $n = (n_1, \ldots, n_q) \in \mathbb{N}^q$,

$$|\Omega_{m_1} + \ldots + \Omega_{m_p} - \Omega_{n_1} - \ldots - \Omega_{n_q}| \geq \frac{\gamma}{\mu_3(m, n)^\alpha}, \qquad (6)$$

  except if the frequencies cancel pairwise. Here, $\mu_3(m, n)$ denotes the third-largest among the integers $m_1, \ldots, m_p$, $n_1, \ldots, n_q$.
- Condition to apply standard Birkhoff normal theorems (Bambusi, Grébert 2006).
- Main difference with NLS (Faou, Gauckler, Lubich 2013)

$$\left| \begin{array}{ll} \Omega_n = \sqrt{n^2 + 2\lambda n} & \text{(logNLS)} \\ \Omega_n = \sqrt{n^2 + 2\lambda \rho^2 n} & \text{(NLS)} \quad \rho = \left\|\psi^0\right\|_{L^2}. \end{array} \right.$$

- **Last change of variable** Matrix S_n^{-1} are real symplectic

$$\begin{pmatrix} \xi_j \\ \overline{\xi}_{-j} \end{pmatrix} = S_n^{-1} \begin{pmatrix} w_j \\ \overline{w}_{-j} \end{pmatrix}, \qquad n = |j|^2 \geq 1$$

New Hamiltonian system

$$i \frac{\mathrm{d}}{\mathrm{d}t} (\xi_j(t)) = \frac{\partial \tilde{H}}{\partial \overline{\xi}_j} (\xi(t), \overline{\xi}(t))$$

$$\tilde{H}(\xi, \overline{\xi}) = H_0 + P = \sum_{j \neq 0} \omega_j |\xi_j|^2 + P(\xi, \overline{\xi}),$$

$$\omega_j = \Omega_n = \sqrt{n^2 + 2\lambda n} \quad \text{for} \quad |j|^2 = n \geq 1$$

P analytic of degree 3 (+ zero momentum condition)

Logarithmic Schrödinger equation

Theorem

(Theorem 7.2 of Grébert 2007)
There exists a canonical transformation $\tau : \mathcal{V} \to \mathcal{U}$ which puts $\tilde{H} = H_0 + P$ in normal form up to order N, i.e.,

$$\tilde{H} \circ \tau = H_0 + Z + R,$$

- $H_0 = \sum_{j \in \mathcal{Z}} \omega_j |\xi_j|^2$,
- *Z is a polynomial of degree N which commutes with all the super-actions, namely $\{Z, J_n\} = 0$ for all $n \geq 1$, $J_n(\xi, \bar{\xi}) = \sum_{|j|^2 = n} |\xi_j|^2$.*
- *$R \in \mathcal{C}^\infty(\mathcal{V}, \mathbb{R})$ and $\left\| X_R(\xi, \bar{\xi}) \right\|_{H^s} \leq C_s \|\xi\|_{H^s}^N$ for $\xi \in \mathcal{V}$,*
- *τ is close to the identity : $\left\| \tau(\xi, \bar{\xi}) - (\xi, \bar{\xi}) \right\|_{H^s} \leq C_s \|\xi\|_{H^s}^2$ for all $\xi \in \mathcal{V}$.*