AP schemes for collisional kinetic equations with fractional diffusion limit

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Notations

- $f(t,x,v) \geq 0$: distribution function of particles
  - $t \geq 0$: time parameter
  - $x \in \mathbb{R}^d$: space parameter
  - $v \in V$: velocity parameter and the symmetric velocity space
- $M(v)$ a given even function such that $\langle M \rangle = 1$

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- $M(v)$ a given even function such that $\langle M \rangle = 1$

$$\begin{cases} M(v) = \frac{m}{1+|v|^\beta}, v \in \mathbb{R}^d \\ \beta \in (d, d+2), \end{cases}$$

- $\varepsilon > 0$ the Knudsen number.

The kinetic equation

$$\begin{cases} \varepsilon^{\alpha}\partial_t f(t,x,v) + \varepsilon v \cdot \nabla_x f(t,x,v) = Lf(t,x,v) \\ f(0,x,v) = f_0(x,v) \\ \text{Periodic boundary conditions,} \end{cases} \tag{1}$$

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In the sequel, we will consider

$$Lf(t,x,v) = \rho(t,x)M(v) - f(t,x,v),$$

with

$$\rho(t,x) = \langle f(t,x,v) \rangle,$$

such that $\langle Lf(t,x,v) \rangle = 0$.

Diffusion asymptotic

When ε goes to 0, the kinetic equation degenerates into a limit equation for ρ. When

$$D = \int_{v \in V} (v \otimes v)\, M(v) \mathrm{d}v < +\infty,$$

the kinetic equation (1) degenerates into a diffusion equation for ρ

$$\begin{cases} \partial_t \rho(t,x) - \nabla_x \cdot (D \nabla_x) \rho(t,x) = 0, \\ \rho(0,x) = \langle f_0(x,v) \rangle . \end{cases}$$

Anomalous diffusion asymptotics

When D is not finite, it is necessary to consider an anomalous scaling

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Then, when $\varepsilon \to 0$, (1) degenerates into a fractional diffusion equation for ρ

$$\begin{cases} \partial_t \rho(t,x) = -C\left(-\Delta_x\right)^{\frac{\alpha}{2}} \rho(t,x) \\ \rho(0,x) = \langle f_0(x,v)\rangle , \end{cases}$$

[2] N. Ben Abdallah, A. Mellet, and M. Puel. "Fractional diffusion limit for collisional kinetic equations : A Hilbert expansion approach". In: *Kinetic and related models* 4.4 (2011), pp. 873–900

[6] A. Mellet, S. Mischler, and C. Mouhot. "Fractional diffusion limit for collisional kinetic equations". In: *Arch. Ration. Mech. Anal.* 199 (2011), pp. 493–525

The fractional laplacian

Can be defined in Fourier variable

$$\left(\widehat{(-\Delta_x)^{\frac{\alpha}{2}}\rho}\right)(k) = |k|^{\alpha}\hat{\rho}(k),$$

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$$\left(\widehat{(-\Delta_x)^{\frac{\alpha}{2}}\rho}\right)(k) = |k|^\alpha \hat{\rho}(k),$$

but has also an integral definition

$$(-\Delta_x)^{\frac{\alpha}{2}}\rho(x) = c_{d,\alpha} P.V. \int_{\mathbb{R}^d} \frac{\rho(x+y)-\rho(x)}{|y|^{d+\alpha}} \mathrm{d}y,$$

it is a non local operator.

AP and UA schemes

- Asympotic Preserving schemes

$$\begin{tikzcd} P_\varepsilon \arrow[r, "\varepsilon\to 0"] & P_0 \\ S_\varepsilon^{\Delta t} \arrow[u, "\Delta t\to 0"] \arrow[r, "\varepsilon\to 0"] & S_0^{\Delta t} \arrow[u, "\Delta t\to 0"] \end{tikzcd}$$

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- Uniform Accuracy (UA)

1 Introduction

2 Anomalous diffusion limit

3 Numerical schemes
- Full implicit scheme
- Micro-macro scheme
- Scheme based on a Duhamel formulation of the equation

4 Case of degenerate collision frequency

Integral formulation in Fourier variable

We can solve (1) in Fourier variable

$$\hat{f}(t,k,v) = \mathrm{e}^{-\frac{t}{\varepsilon^\alpha}(1+\mathrm{i}\varepsilon k\cdot v)}\hat{f}_0(k,v) + \int_0^{\frac{t}{\varepsilon^\alpha}} M(v)\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)}\hat{\rho}(t-\varepsilon^\alpha s,k)\mathrm{d}s,$$

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$$\hat{\rho}(t,k) = \left\langle \mathrm{e}^{-\frac{t}{\varepsilon^\alpha}(1+\mathrm{i}\varepsilon k\cdot v)}\hat{f}_0(k,v)\right\rangle + \int_0^{\frac{t}{\varepsilon^\alpha}} \left\langle M(v)\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)}\right\rangle \hat{\rho}(t-\varepsilon^\alpha s,k)\mathrm{d}s.$$

Limit for $\hat{f}$

The equation for $\hat{f}$ leads to

$$\hat{f}(t,x,v) = \int_0^\infty M(v)\hat{\rho}(t-\varepsilon^\alpha s, k)\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)}\mathrm{d}s,$$

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indeed

$$\hat{f}(t,k,v) = \hat{\rho}(t,k)M(v) + o(1).$$

Limit for $\hat{\rho}$

The equation for $\hat{\rho}$ can be rewrited as

$$\begin{aligned}\hat{\rho}(t,k) =& \left\langle \mathrm{e}^{-\frac{t}{\varepsilon^\alpha}(1+\mathrm{i}\varepsilon k\cdot v)}\hat{f}_0(k,v)\right\rangle \\ &+ \varepsilon^\alpha \int_0^{\frac{t}{\varepsilon^\alpha}} s\left\langle M(v)\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)}\right\rangle \frac{\hat{\rho}(t-\varepsilon^\alpha s,k)-\hat{\rho}(t,k)}{\varepsilon^\alpha s}\mathrm{d}s \\ &+ \int_0^{\frac{t}{\varepsilon^\alpha}} \left\langle M(v)\left(\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)} - \mathrm{e}^{-s}\right)\right\rangle \mathrm{d}s\hat{\rho}(t,k) \\ &+ \int_0^{\frac{t}{\varepsilon^\alpha}} \left\langle M(v)\mathrm{e}^{-s}\right\rangle \mathrm{d}s\hat{\rho}(t,k),\end{aligned}$$

Limit for $\hat{\rho}$

that is

$$\begin{aligned}
\hat{\rho}(t,k) &= o(\varepsilon^{\infty}) \\
&+ \varepsilon^{\alpha} \int_0^{\frac{t}{\varepsilon^{\alpha}}} s \left\langle M(v) \mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)} \right\rangle \frac{\hat{\rho}(t-\varepsilon^{\alpha}s,k) - \hat{\rho}(t,k)}{\varepsilon^{\alpha}s} \mathrm{d}s \\
&+ \int_0^{\frac{t}{\varepsilon^{\alpha}}} \left\langle M(v) \left(\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)} - \mathrm{e}^{-s} \right) \right\rangle \mathrm{d}s \hat{\rho}(t,k) \\
&+ \int_0^{\frac{t}{\varepsilon^{\alpha}}} \left\langle M(v) \mathrm{e}^{-s} \right\rangle \mathrm{d}s \hat{\rho}(t,k),
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&+ \hat{\rho}(t,k) + o(\varepsilon^\infty),
\end{aligned}$$

Making the anomalous diffusion appear

$$\lim_{\varepsilon \to 0} \int_0^{\frac{t}{\varepsilon^\alpha}} \left\langle M(v) \left(\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k \cdot v)} - \mathrm{e}^{-s} \right) \right\rangle \mathrm{d}s \ ?$$

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We perform a change of variables in the integral in v

$$w = \varepsilon |k| v,$$

and it becomes

$$(\varepsilon|k|)^{\beta-d} \int_0^{\frac{t}{\varepsilon^\alpha}} \left\langle \left(\mathrm{e}^{-s(1+\mathrm{i}w\cdot e)} - \mathrm{e}^{-s} \right) \frac{m}{(\varepsilon|k|)^\beta + |w|^\beta} \right\rangle \mathrm{d}s.$$

where e is any unitary vector.

Limit equation

Proposition

When $\varepsilon \to 0$, $\hat{\rho}$ *solves the anomalous diffusion equation*

$$\partial_t \hat{\rho}(t,k) = -\kappa |k|^{\alpha} \hat{\rho}(t,k),$$

with

$$\begin{cases} \kappa = \left\langle \frac{(v\cdot e)^2}{1+(v\cdot e)^2} \frac{m}{|v|^{d+\alpha}} \right\rangle \\ \alpha = \beta - d \end{cases}.$$

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A similar derivation can be done in the space variable.

We consider a fully implicit time discretization of (1)

$$\frac{\hat{f}^{n+1} - \hat{f}^{n}}{\Delta t} + \frac{1}{\varepsilon^{\alpha}} \left(1 + \mathrm{i}\varepsilon k \cdot v\right) \hat{f}^{n+1} = \frac{1}{\varepsilon^{\alpha}} \hat{\rho}^{n+1} M(v),$$

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that is

$$\hat{f}^{n+1} = \frac{1-\lambda}{1 + \mathrm{i}\lambda\varepsilon k \cdot v} \hat{f}^{n} + \frac{\lambda}{1 + \mathrm{i}\lambda\varepsilon k \cdot v} \hat{\rho}^{n+1} M(v),$$

where $\lambda = \frac{\Delta t}{\varepsilon^{\alpha} + \Delta t} \in [0, 1]$.

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where $\lambda = \frac{\Delta t}{\varepsilon^\alpha + \Delta t} \in [0,1]$. As $\lambda \underset{\varepsilon \to 0}{\longrightarrow} 1$ and $\lambda \underset{\Delta t \to 0}{\longrightarrow} 0$

$$\hat{f}^{n+1} = \hat{\rho}^{n+1} M(v) + o(1),$$

for small ε.

The expression for $\hat{f}^{n+1}$ gives an expression for $\hat{\rho}^{n+1}$ after an integration in v

$$\hat{\rho}^{n+1} = \frac{\left\langle \dfrac{\frac{1}{\varepsilon^\alpha + \Delta t} \hat{f}^n}{1 + \mathrm{i}\varepsilon k \cdot v} \right\rangle}{\left\langle \dfrac{\frac{1}{\varepsilon^\alpha + \Delta t} M(v)}{1 + \mathrm{i}\lambda\varepsilon k \cdot v} \right\rangle + \varepsilon^{2-\alpha} \left\langle \dfrac{M(v)\lambda^2 \left(k \cdot v\right)^2}{1 + \lambda^2 \left(\varepsilon k \cdot v\right)^2} \right\rangle},$$

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but in numerical computations, the limit scheme is

$$\hat{\rho}^{n+1} = \hat{\rho}^n.$$

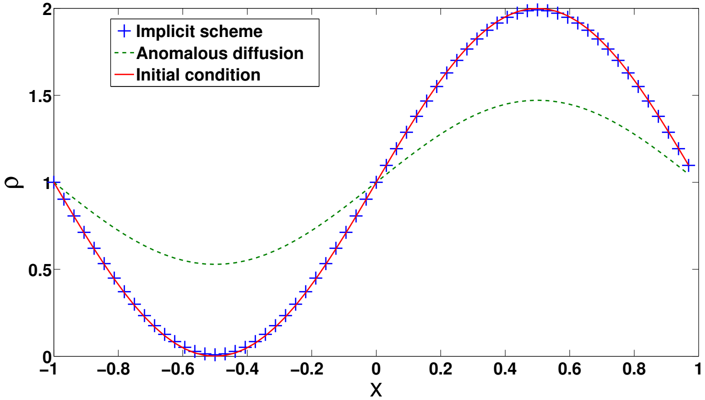

Figure : For $\Delta t = 10^{-3}$, the densities $\rho(t = 0.1, x)$ for $\varepsilon = 10^{-6}$.

Modification of the integral giving the limit equation

We apply the change of variables $w = \varepsilon\lambda|k|v$

$$\mathcal{I} = |k|^\alpha \lambda^\alpha \left\langle \frac{m}{(\varepsilon\lambda|k|)^{d+\alpha} + |w|^{d+\alpha}} \frac{(w \cdot e)^2}{1 + (w \cdot e)^2} \right\rangle,$$

where e is any unitary vector.

Proposition (N. Crouseilles, H. H., M. Lemou - [4])

We consider the following scheme defined for all k and for all time index $0 \leq n \leq N, N\Delta t = T$ by

$$\begin{cases} \hat{f}^{n+1} = \left((1-\lambda)\hat{f}^n + \lambda\hat{\rho}^{n+1}M(v)\right)\left(1 + \mathrm{i}\lambda\varepsilon k \cdot v\right)^{-1} \\ \hat{\rho}^{n+1} = \dfrac{\left\langle \dfrac{\frac{1}{\varepsilon^\alpha + \Delta t}\hat{f}^n}{1 + \mathrm{i}\lambda\varepsilon k \cdot v} \right\rangle}{\left\langle \dfrac{\frac{1}{\varepsilon^\alpha + \Delta t}M(v)}{1 + \mathrm{i}\lambda\varepsilon k \cdot v} \right\rangle + \mathcal{I}}. \end{cases}$$

1. *The scheme is of order 1 for any fixed ε and preserves the total mass.*
2. *The scheme is AP : for a fixed Δt, the scheme solves the anomalous diffusion equation when ε goes to zero*

$$\frac{\hat{\rho}^{n+1}(k) - \hat{\rho}^n(k)}{\Delta t} = -\kappa|k|^\alpha \hat{\rho}^{n+1}(k).$$

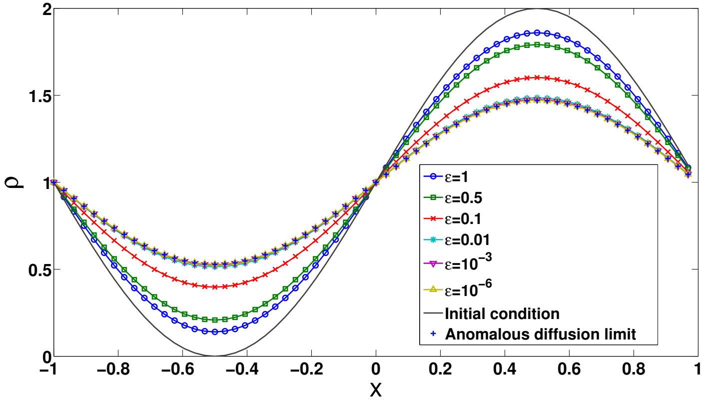

Figure : For $\Delta t = 10^{-3}$, the densities $\rho(t = 0.1, x)$ for a range of ε.

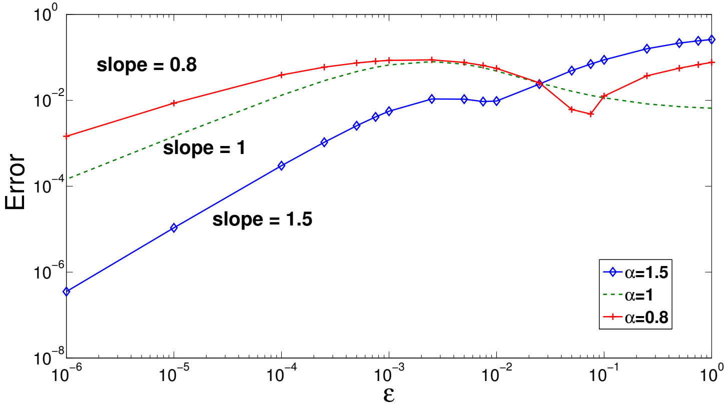

Figure : For $\Delta t = 10^{-3}$, the error as a function of ε.

The semi-discrete scheme

Set $f = \rho M + g$ such that $\rho = \langle f \rangle$ and $\langle g \rangle = 0$

$$\partial_t \rho + \varepsilon^{1-\alpha} \left\langle v \cdot \nabla_x g \quad \right\rangle = 0$$

$$\partial_t g + \varepsilon^{1-\alpha} v \cdot \nabla_x \rho \; M + \varepsilon^{1-\alpha} \left(v \cdot \nabla_x g \; - \langle v \cdot \nabla_x g \; \rangle M(v) \right)$$

$$= -\frac{1}{\varepsilon^\alpha} g \qquad .$$

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$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \varepsilon^{1-\alpha} \left\langle v \cdot \nabla_x g^{n+1} \right\rangle = 0$$

$$\frac{g^{n+1} - g^n}{\Delta t} + \varepsilon^{1-\alpha} v \cdot \nabla_x \rho^n M + \varepsilon^{1-\alpha} \left(v \cdot \nabla_x g^n - \langle v \cdot \nabla_x g^n \rangle M(v) \right)$$
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$$= -\frac{1}{\varepsilon^\alpha} g^{n+1}.$$

For small ε

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} - \varepsilon^{2-\alpha} \left\langle v \cdot \nabla_x \left(v \cdot \nabla_x \rho^n M(v) \right) \right\rangle = 0.$$

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For small ε

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} - \varepsilon^{2-\alpha} \left\langle v \cdot \nabla_x \left(v \cdot \nabla_x \rho^n M(v) \right) \right\rangle = 0.$$

We consider an implicit formulation of (1)

$$f^{n+1} = \lambda \left(I + \varepsilon\lambda v \cdot \nabla_x\right)^{-1} \rho^{n+1} M(v)$$
$$+ (1-\lambda) \left(I + \varepsilon\lambda v \cdot \nabla_x\right)^{-1} f^n,$$

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$$f^{n+1} = \lambda \left(I + \varepsilon\lambda v \cdot \nabla_x\right)^{-1} \rho^{n+1} M(v) \\ + (1-\lambda)\left(I - \varepsilon\lambda v \cdot \nabla_x\right) f^n,$$

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and we inject it in the first line of the scheme

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \frac{1}{\varepsilon^\alpha} \left\langle \varepsilon\lambda v \cdot \nabla_x \left(I + \varepsilon\lambda v \cdot \nabla_x\right)^{-1} \rho^{n+1} M(v) \right\rangle + \varepsilon^{1-\alpha}(1-\lambda)\left\langle v \cdot \nabla_x g^n \right\rangle = 0.$$

The scheme writes

$$\begin{cases} \dfrac{g^{n+1}-g^n}{\Delta t} + \mathcal{T}^n = -\dfrac{1}{\varepsilon^\alpha} g^{n+1} \\ \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \mathcal{F}^{-1}\left(\mathcal{I}\hat{\rho}^{n+1/2}\right) + \varepsilon^{1-\alpha}(1-\lambda)\langle v\cdot\nabla_x g^n\rangle = 0, \end{cases}$$

with

$$\mathcal{T}^n = \varepsilon^{1-\alpha} v\cdot\nabla_x \rho^n M(v) + \varepsilon^{1-\alpha}\left(v\cdot\nabla_x g^n - \langle v\cdot\nabla_x g^n\rangle M(v)\right)$$

and

$$\mathcal{I} = \varepsilon^{-\alpha}\left\langle \frac{\lambda^2(\varepsilon k\cdot v)^2 M(v)}{1+\lambda^2(\varepsilon k\cdot v)^2}\right\rangle_{\text{modified}}.$$

Proposition (N. Crouseilles, H. H., M. Lemou, [3]-[4])

We consider the previous semi-discrete scheme defined for all $x \in \mathbb{R}^d, v \in B(0,\delta) \subset \mathbb{R}^d$ and all time index $0 \leq n \leq N$ with $N\Delta t = T(T > 0)$, with initial conditions

$$\begin{cases} \rho^0(x) = \rho(0,x) \\ g^0(x,v) = f_0(x,v) - \rho(0,x)M(v). \end{cases}$$

This scheme has the following properties:

1. *The scheme is of order 1 for any fixed ε and preserves the total mass .*
2. *The scheme is AP : for a fixed Δt, the scheme solves the anomalous diffusion equation when ε goes to zero*

$$\frac{\hat{\rho}^{n+1}(k) - \hat{\rho}^n(k)}{\Delta t} = -\kappa|k|^\alpha \hat{\rho}^{n+1/2}(k).$$

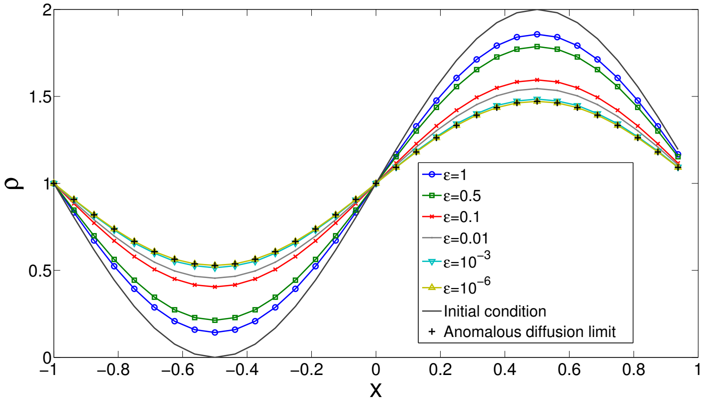

Figure : For $\Delta t = 10^{-3}$ and $\alpha = 1.5$, the densities for a range of ε.

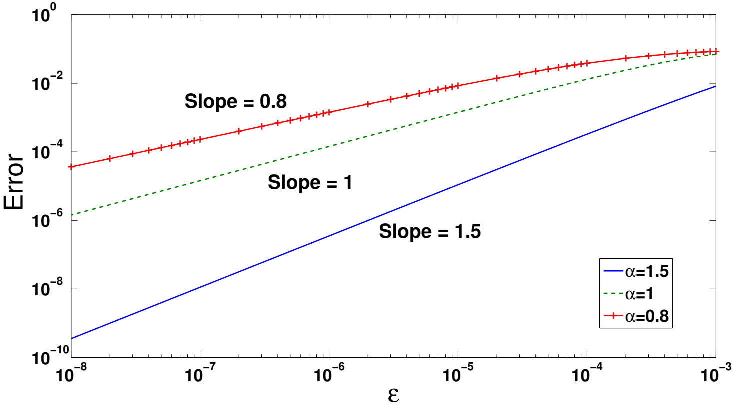

Figure : For $\Delta t = 10^{-3}$ the error as a function of ε

Semi-discrete scheme

The integral formulation of (1) is

$$\hat{\rho}(t,k) = \left\langle \mathrm{e}^{-\frac{t}{\varepsilon^\alpha}(1+\mathrm{i}\varepsilon k\cdot v)} \hat{f}_0(k,v) \right\rangle + \int_0^{\frac{t}{\varepsilon^\alpha}} \left\langle M(v)\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)} \right\rangle \hat{\rho}(t-\varepsilon^\alpha s, k)\mathrm{d}s,$$

Semi-discrete scheme

that is

$$\hat{\rho}(t_{n+1},k) = \hat{A}_0(t_{n+1},k)$$
$$+\sum_{j=0}^{n}\int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left\langle M(v)\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)}\right\rangle \hat{\rho}(t_{n+1}-\varepsilon^\alpha s,k)\mathrm{d}s.$$

Semi-discrete scheme

that is

$$\hat{\rho}(t_{n+1},k) = \hat{A}_0(t_{n+1},k) + \sum_{j=0}^{n} \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left\langle M(v) \mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)} \right\rangle \hat{\rho}(t_{n+1}-\varepsilon^\alpha s,k)\mathrm{d}s.$$

We have

$$\hat{\rho}(t_{n+1}-\varepsilon^\alpha s,k) = \left(1-\frac{\varepsilon^s - t_j}{\Delta t}\right)\hat{\rho}(t_{n+1}-t_j,k) + \frac{\varepsilon^s - t_j}{\Delta t}\hat{\rho}(t_{n+1}-t_{j+1},k) + O\left(\Delta t^2\right)$$

The scheme writes

$$\hat{\rho}^{n+1}(k) = \frac{\hat{A}_0(t_{n+1},k) + \sum_{j=1}^{n}\left(c_j(\varepsilon,k)\hat{\rho}^{n+1-j}(k) + b_j(\varepsilon,k)\hat{\rho}^{n-j}(k)\right) + b_0(\varepsilon,k)\hat{\rho}^n(k)}{1 - c_0(\varepsilon,k)}.$$

with

$$b_j(\varepsilon,k) = \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s - t_j}{\Delta t}\left\langle M(v)\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)}\right\rangle \mathrm{d}s$$

$$c_j(\varepsilon,k) = \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t}\right)\left\langle M(v)\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)}\right\rangle \mathrm{d}s.$$

The scheme writes

$$\hat{\rho}^{n+1}(k) = \frac{\hat{A}_0(t_{n+1},k) + \sum_{j=1}^{n} \left(c_j(\varepsilon,k)\hat{\rho}^{n+1-j}(k) + b_j(\varepsilon,k)\hat{\rho}^{n-j}(k)\right) + b_0(\varepsilon,k)\hat{\rho}^n(k)}{1-c_0(\varepsilon,k)}.$$

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A similar derivation gives a scheme for $\hat{f}$.

Modification of b_j and c_j

We rather use the expressions

$$b_j(\varepsilon,k) = \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s - t_j}{\Delta t} \left\langle M(v)\mathrm{e}^{-s}\right\rangle \mathrm{d}s + \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \frac{\varepsilon^\alpha s - t_j}{\Delta t} \left\langle M(v)\left(\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)} - \mathrm{e}^{-s}\right)\right\rangle_{\sim} \mathrm{d}s$$

$$c_j(\varepsilon,k) = \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t}\right) \left\langle M(v)\mathrm{e}^{-s}\right\rangle \mathrm{d}s + \int_{\frac{t_j}{\varepsilon^\alpha}}^{\frac{t_{j+1}}{\varepsilon^\alpha}} \left(1 - \frac{\varepsilon^\alpha s - t_j}{\Delta t}\right) \left\langle M(v)\left(\mathrm{e}^{-s(1+\mathrm{i}\varepsilon k\cdot v)} - \mathrm{e}^{-s}\right)\right\rangle_{\sim} \mathrm{d}s.$$

with the time integrations done exactly.

Proposition (N. Crouseilles, H. H., M. Lemou, [3]-[4])

We consider the previous scheme defined for all k and for all time index $0 \leq n \leq N, N\Delta t = T$. For the degenerate collision frequency case, we consider only initial conditions at equilibrium.
This scheme has the following properties:

1. *The scheme preserves the total mass and is of order 1 uniformly in ε*

$$\exists C > 0, \sup_{\varepsilon \in (0,1]} \|\hat{\rho}^N(k) - \hat{\rho}(T,k)\| \leq C\Delta t.$$

2. *The scheme is AP: for a fixed Δt, the scheme solves the anomalous diffusion equation when ε goes to zero*

$$\frac{\hat{\rho}^{n+1}(k) - \hat{\rho}^n(k)}{\Delta t} = -\kappa|k|^\alpha \hat{\rho}^{n+1}(k).$$

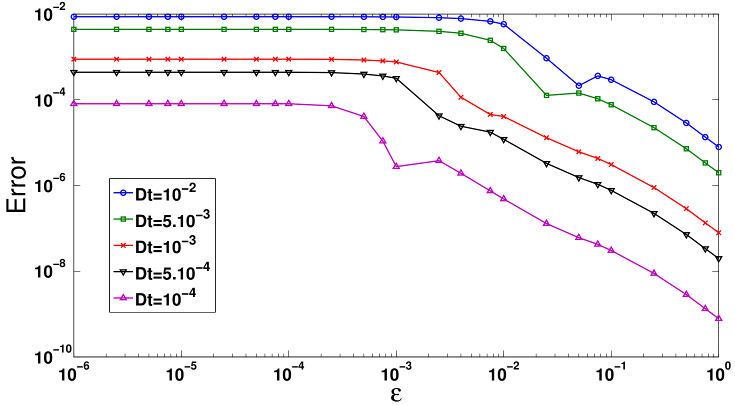

Figure : For $\alpha = 1.5$, the error computed for a range of Δt as functions of ε.

- [3] N. Crouseilles, H. Hivert, and M. Lemou. "Multiscale numerical schemes for kinetic equations in the anomalous diffusion limit". In: *Comptes Rendus Mathematique* 353.8 (2015), pp. 755 –760. issn: 1631-073X
- [4] N. Crouseilles, H. Hivert, and M. Lemou. "Numerical schemes for kinetic equations in the diffusion and anomalous diffusion limit. Part I: the case of heavy-tailed equilibrium". In: *SIAM, J. Sc. Comput.* (2016)

A similar phenomenon occurs in the case of a degenerate collision frequency. Namely, we consider :

$$\begin{cases} M(v) = m, & |v| \leq 1 \\ \nu(v) = \nu_0 |v|^{d+2+\beta} & |v| \leq 1, \beta > 0, \end{cases}$$

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and the kinetic equation

$$\varepsilon^{\alpha} \partial_t f + \varepsilon v \cdot \nabla_x f = \nu \left(\rho_\nu M - f \right),$$

with

$$\rho_\nu = \frac{\langle \nu f \rangle}{\langle \nu M \rangle}.$$

Fractional diffusion limit

Because of the low velocities,

$$D = \int_{|v|\leq 1} \frac{v \otimes v}{\nu(v)} M(v) \mathrm{d}v = +\infty.$$

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The use of an anomalous scaling $\alpha = \frac{2d+2+\beta}{d+1+\beta} \in (1,2)$, and of the change of variables $w = \frac{\varepsilon|k|v}{\nu(v)}$, leads to a fractional diffusion equation (here in Fourier variable)

$$\partial_t \hat{\rho}(t,k) = -\kappa|k|^\alpha \hat{\rho}(t,k).$$

[1] N. Ben Abdallah, A. Mellet, and M. Puel. "Anomalous diffusion limit for kinetic equations with degenerate collision frequency". In: *Math. Models Methods Appl. Sci.* 21.11 (2011), pp. 2249–2262

- Use of similar strategy to deal with the low velocities.

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- Additional difficulties arise :
  - Dealing with the renormalized density ρ_ν
  - Slow convergence towards the small ε limit :

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- Three numerical schemes with similar properties
  - Implicit in Fourier variable scheme
  - Micro-macro scheme
  - Duhamel formulation based scheme

[5] N. Crouseilles, H. Hivert, and M. Lemou. "Numerical schemes for kinetic equations in the diffusion and anomalous diffusion limit. Part II: the case of degenerate collision frequency". In: *hal-01245312* (2015)

Convergence rate towards the fractional diffusion limit

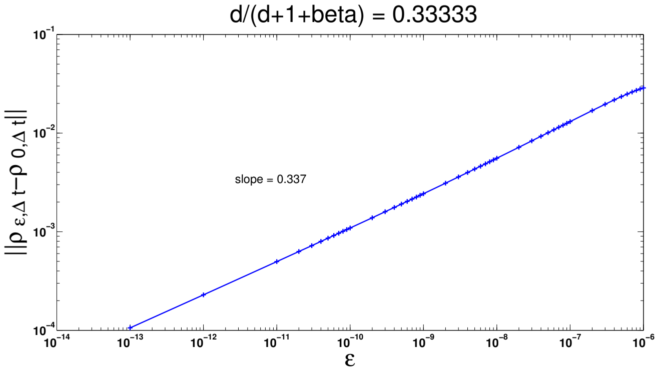

Figure : For $\beta = 1$ and $\Delta t = 10^{-4}$, the convergence towards the limit equation as a function of ε (MM scheme).

Conclusion

Future works :

- Study of the limit case $\beta = d + 2$
- Deal with more general collision operators
- Boundary conditions

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Thank you for your attention !