Inégalités de dispersion via le semi-groupe de la chaleur

Valentin Samoyeau,

Advisor: Frédéric Bernicot.
Laboratoire de Mathématiques Jean Leray, Université de Nantes

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$$\begin{cases} i\partial_t u + \Delta u = F(u) \\ u(0,x) = u_0(x) \end{cases}, \quad t \in \mathbb{R},\ x \in \mathbb{R}^d. \tag{NLS}$$

- Duhamel's formula:
  $u(t,x) = e^{it\Delta}u_0(x) - i\int_0^t e^{i(t-s)\Delta}F(u(s,x))ds.$
- Existence, uniqueness: Contraction principle.
  Relies on Strichartz estimates: $\forall\ 2 \leq p, q \leq +\infty$

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} \Rightarrow \|e^{it\Delta}u_0\|_{L_t^p L_x^q} \lesssim \|u_0\|_{L^2}. \tag{1}$$

- Via a TT^* argument, interpolation with $\|e^{it\Delta}\|_{L^2\to L^2} \lesssim 1$, and Hardy-Littlewood-Sobolev inequality (Keel-Tao), (1) reduces to $L^1 - L^\infty$ dispersion inequality:

$$\|e^{it\Delta}\|_{L^1\to L^\infty} \lesssim |t|^{-\frac{d}{2}}. \tag{2}$$

- (2) can be obtained by a complexification of the heat semigroup $(e^{t\Delta})_{t\geq 0}$.
- In $\mathbb{R}^d$ we have an explicit formulation of the heat semigroup kernel:

$$p_t(x,y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{4t}}.$$

- Strichartz estimates with loss of derivatives

$$\|e^{it\Delta}u_0\|_{L^p_t L^q_x} \lesssim \|u_0\|_{W^{s,2}}.$$

- Local-in-time Strichartz estimates

$$\|e^{it\Delta}u_0\|_{L^p(t\in[0,T],L^q_x)} \lesssim \|u_0\|_{W^{s,2}}.$$

Question:

What do we know outside of $\mathbb{R}^d$ with the usual Laplacian Δ ?

Examples

- Outside of a smooth convex domain of $\mathbb{R}^d$ with Laplace-Beltrami operator: global-in-time estimates with loss of $\frac{1}{p}$ derivatives [Burq-Gérard-Tzvetkov].
- Compact riemannian manifold: local-in-time estimates with loss of $\frac{1}{p}$ derivatives [Burq-Gérard-Tzvetkov].
- Asymptotically hyperbolic manifolds: local-in-time estimates without loss [Bouclet].
- Laplacian with a smooth potential, infinite manifolds with boundary with one trapped orbit: local-in-time estimates with $\frac{1}{p} + \varepsilon$ loss of derivatives [Christianson].

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Remark: One cannot expect global-in-time estimates in a compact setting.
Example of a constant initial data for the Torus: $t \nrightarrow +\infty$

$$\|e^{it\Delta}u_0 = u_0\|_{L^\infty(\mathbb{T})} \leq C|t|^{-\frac{d}{2}}\|u_0\|_{L^1(\mathbb{T})} \Leftrightarrow 1 \lesssim |t|^{-\frac{d}{2}}.$$

Theorem [Burq-Gérard-Tzvetkov, '04]

Let $\mathcal{M}$ be a compact riemannian manifold of dimension d. If $\varphi \in C_0^\infty(\mathbb{R}_+)$ then for all $h \in]0,1]$:

$$\|e^{it\Delta}\varphi(h^2\Delta)\|_{L^1\to L^\infty} \lesssim |t|^{-\frac{d}{2}}, \quad |t| \lesssim h.$$

- Example of the sphere $\mathbb{S}^3$: optimal loss of $\frac{1}{p}$ ([BGT]);
- By Sobolev embeddings, the condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

gives a straightforward loss of $\frac{2}{p}$.

Conclusion: The loss γ is interesting when $\gamma \leq \frac{2}{p}$.

1. Introduction
   - Schrödinger's equation
   - Strichartz estimates in various settings
2. Framework
   - Space of homogeneous type
   - Heat semigroup
   - Hardy and BMO spaces
3. Results
   - Reduction to L2-L2 estimates
   - From wave dispersion to Schrödinger dispersion
   - Applications
   - Weak wave dispersion
4. Conclusion
   - Perspectives

The space:
(X, d, μ) is a metric measured space with μ satisfying a doubling property:

$$\forall x \in X,\ \forall r > 0,\ \mu(B(x, 2r)) \leq C\mu(B(x, r)). \tag{3}$$

Then there exists a homogeneous dimension d such that:

$$\forall x \in X, \forall r > 0, \forall \lambda \geq 1, \mu(B(x, \lambda r)) \lesssim \lambda^d \mu(B(x, r)).$$

Examples

Euclidean space $\mathbb{R}^d$, open sets of $\mathbb{R}^d$, smooth manifolds of dimension d, some fractals sets, Lie groups, Heisenberg group,...

The operator:

- H is a self-adjoint nonnegative operator, densely defined on $L^2(X)$.
- H generates a L^2-holomorphic semigroup $(e^{-tH})_{t\geq 0}$ (Davies).
- The evolution problem we study is

$$\begin{cases} i\partial_t u + Hu = F(u) \\ u(0,x) = u_0(x) \end{cases}, \ x \in X.$$

Remark: Semigroup structure $\Rightarrow \psi_m : x \mapsto x^m e^{-x}$ are easier to handle than $\varphi \in C_0^\infty$.

- Typical on-diagonal upper estimates:

$$\forall t>0,\ \forall x\in X,\ 0\leq p_t(x,x)\lesssim \frac{1}{\mu(B(x,\sqrt{t}))} \qquad \text{(DUE)}$$

- Self-improve (Coulhon-Sikora) into full gaussian estimates:

$$\forall t>0,\ \forall x,y\in X,\ 0\leq p_t(x,y)\lesssim \frac{1}{\mu(B(x,\sqrt{t}))}e^{-\frac{d(x,y)^2}{4t}}. \qquad \text{(UE)}$$

- Davies-Gaffney estimates:

$$\forall t>0,\ \forall E,F\subset X,\ \|e^{-tH}\|_{L^2(E)\to L^2(F)}\lesssim e^{-\frac{d(E,F)^2}{4t}} \qquad \text{(DG)}$$

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Some cases where the previous estimates hold:

Examples

- (DUE): Δ on a domain with boundary conditions, semigroup generated by a self-adjoint operator of divergence form $H = -\mathrm{div}(A\nabla)$ with A a real bounded elliptic matrix on $\mathbb{R}^d$;
- (UE): $H = -\sum_{i=1}^{d} X_i^2$ where X_i are vector fields satisfying Hörmander condition on a Lie group or a riemannian manifold with bounded geometry;
- (DG): most second order self-adjoint differential operators, Laplace-Beltrami on a riemannian manifold, Schrödinger operator with potential...

- **Question:** how to prove $L^1 - L^\infty$ dispersive estimates

$$\|e^{itH}\psi_m(h^2H)\|_{L^1(X)\to L^\infty(X)} \lesssim |t|^{-\frac{d}{2}} \quad ?$$

- **Answer:** I don't know...
- Instead we prove $H^1 - \mathrm{BMO}$ estimates.
- **Remark:** The classical Hardy space (of Coifman-Weiss) H^1 and BMO (of John-Nirenberg) are not adapted to the semigroup setting;
- We use an abstract construction of Bernicot-Zhao to construct equivalent spaces.

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- For a ball Q of radius $r > 0$ we set

$$B_Q = (\mathrm{Id} - e^{-r^2 H})^M \simeq \sum_{k=0}^{M} e^{-kr^2 H};$$

- a is an atom associated with the ball Q if there is f_Q supported in Q with $\|f_Q\|_{L^2(Q)} \leq \mu(Q)^{-\frac{1}{2}}$ such that

$$a = B_Q(f_Q);$$

- $$h \in H^1 \Leftrightarrow h = \sum_i \lambda_i a_i$$

  where a_i are atoms and $\sum_i |\lambda_i| < +\infty$;
- $\|h\|_{H^1} := \inf\{\sum_i |\lambda_i|,\ h = \sum_i \lambda_i a_i\}$.

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- If $f \in L^\infty$ we set

$$\|f\|_{\mathrm{BMO}} = \sup_Q \left(\fint_Q |B_Q(f)|^2 d\mu \right)^{\frac{1}{2}} ;$$

- The space BMO is defined as the closure

$$\mathrm{BMO} := \overline{\{f \in L^\infty + L^2, \ \|f\|_{\mathrm{BMO}} < +\infty\}},$$

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The key property of H^1 and BMO is the interpolation theorem

Theorem (Bernicot, '09)

For all $\theta \in (0,1)$, using interpolation notations we have

$$(L^2, H^1)_\theta = L^p \quad \text{and} \quad (L^2, \mathrm{BMO})_\theta \hookrightarrow L^q.$$

with $p \in (1,2)$ and $q = p' \in (2, \infty)$ given by

$$\frac{1}{p} = \frac{1-\theta}{2} + \theta \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{2}.$$

The question we investigate is how to prove $H^1 - \mathrm{BMO}$ dispersive estimates

$$\|e^{itH}\psi_m(h^2H)\|_{H^1\to\mathrm{BMO}} \lesssim |t|^{-\frac{d}{2}}.$$

Remark: $e^{itH}\psi_m(h^2H) = (h^2H)^m e^{-zH}$ with $z = h^2 - it$.

- $|t| \leq 1$ (i.e. t independant of h) is difficult.
- $|t| \leq h^2$ is straightforward by analytic continuation of (UE) (since $\mathrm{Re}(z) \simeq |z| \geq |t|$).
- $h^2 \leq |t| \leq h$ is dealt by [BGT, '04] in the compact riemannian manifold setting (using pseudo-differential tools).
- We will treat the case $h^2 \leq |t| \leq h^{1+\varepsilon}$ (for all $\varepsilon > 0$).

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Hypothesis ($H_m(A)$)

An operator T satisfies Hypothesis ($H_m(A)$) if:

$$\forall r > 0, \ \|T\psi_m(r^2 H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim A\mu(B)^{\frac{1}{2}}\mu(\widetilde{B})^{\frac{1}{2}}, \qquad (H_m(A))$$

for any two balls B, $\widetilde{B}$ of radius r.

Remarks:

- We intend to use hypothesis ($H_m(A)$) for $T = e^{itH}\psi_m(h^2 H)$ and $A = |t|^{-\frac{d}{2}}$.
- Hypothesis ($H_m(A)$) is weaker than the $L^1 - L^\infty$ estimate by Cauchy-Schwarz inequality.

Theorem 1 [Bernicot, S., '14]

Let T be a self-adjoint operator commuting with H. If T satisfies $(H_m(A))$ for $m \geq \frac{d}{2}$, then

$$\|T\|_{H^1 \to \mathrm{BMO}} \lesssim A.$$

That theorem reduces the $H^1 - \mathrm{BMO}$ estimates to microlocalized $L^2(B) - L^2(\widetilde{B})$ ones.

Moreover, if $\|T\|_{L^2 \to L^2} \lesssim 1$ then we can interpolate to get

$$\|T\|_{L^p \to L^{p'}} \lesssim A^{\frac{1}{p} - \frac{1}{p'}}.$$

Ideas of the proof:

- Use the atomic structure of H^1.
- Use an approximation of the identity well suited to our setting $(e^{-sH})_{s>0}$.

Summary of theorem 1

$(H_m(A)) \Rightarrow H^1 \to \mathrm{BMO}$ and $L^p \to L^{p'}$ dispersive estimates.

Wave propagator

For $f \in L^2$, we note $\cos(t\sqrt{H})f$ the unique solution at time t of the wave problem:

$$\begin{cases} \partial_t^2 u + Hu = 0 \\ u_{|t=0} = f \\ \partial_t u_{|t=0} = 0 \end{cases}.$$

The wave propagator is the map $f \mapsto \cos(t\sqrt{H})f$.

Finite speed propagation

For any disjoint open sets $U_1, U_2 \subset X$, and any $f_1 \in L^2(U_1)$, $f_2 \in L^2(U_2)$, we have:

$$\forall 0 < t < d(U_1, U_2), \ < \cos(t\sqrt{H}) f_1, f_2 >= 0. \tag{4}$$

We have the equivalence (Coulhon-Sikora '06):

$$(\mathrm{DG}) \Leftrightarrow (4).$$

Remark: If $\cos(t\sqrt{H})$ has a kernel K_t, (4) means that K_t is supported in the "light cone":

$$\mathrm{supp}\ K_t \subset \{(x, y) \in X^2, \ d(x, y) \leq t\}.$$

Assumption on the wave propagator

There exists $\kappa \in (0, \infty]$ and an integer ℓ such that for all $s \in (0, \kappa)$, for all $r > 0$ and any two balls B, $\widetilde{B}$ of radius r

$$\| \cos(s\sqrt{H})\psi_\ell(r^2 H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}} \left(\frac{r}{r+|L-s|}\right)^{\frac{d+1}{2}}$$

where $L = d(B, \widetilde{B})$.

Remark: κ is linked to the geometry of the space X (its injectivity radius for example).

Theorem 2 [Bernicot, S. '14]

Under the previous assumption on the wave propagator, for all $m \geq \max\{\frac{d}{2}, \ell + \lceil \frac{d-1}{2} \rceil\}$:

1. If $\kappa = +\infty$: e^{itH} satisfies $(H_m(|t|^{-\frac{d}{2}}))$ for all $t \in \mathbb{R}$.
2. If $\kappa < +\infty$: for all $\varepsilon > 0$ and $h > 0$ with $|t| \leq h^{1+\varepsilon}$ and all integer $m' \geq 0$, $e^{itH}\psi_{m'}(h^2 H)$ satisfies $(H_m(|t|^{-\frac{d}{2}}))$.

- In the first case we obtain global-in-time Strichartz estimates without loss of derivatives.
- In the second case we recover local-in-time Strichartz estimates with $\frac{1}{p} + \varepsilon$ loss of derivatives.

Ideas of the proof:

- Cauchy formula $\Rightarrow e^{-zH} = \int_0^{+\infty} \cos(s\sqrt{H}) e^{-\frac{s^2}{4z}} \frac{ds}{\sqrt{\pi z}}$ with $z = h^2 - it$;
- Integrate by parts when s is small;
- Use assumption on $\cos(s\sqrt{H})$ when $s < \kappa$;
- Use the exponential decay of $e^{-\frac{s^2}{4z}}$ when s is large.

Summary of theorem 2

$L^2(B) \to L^2(\widetilde{B})$ dispersion for the wave propagator $\Rightarrow H_m(|t|^{-\frac{d}{2}})$.

Some cases where we can check $L^2(B) \to L^2(\widetilde{B})$ dispersion for the wave propagator to apply Theorem 1 and 2 and recover Strichartz estimates:

Examples

- $X = \mathbb{R}^d$ with $H = -\Delta$ $(\kappa = +\infty)$;
- $X = \mathbb{R}^d$ with $H = -\mathrm{div}(A\nabla)$ where $A \in C^{1,1}$ $(\kappa < +\infty)$;
- Compact riemannian manifolds with Laplace-Beltrami operator (κ depends on the injectivity radius);
- Non-compact riemannian manifolds with bounded geometry (κ given by the geometry);
- Non-trapping asymptotically conic manifolds with $H = -\Delta + V$ ([Hassel-Zhang '15]).

For the Laplacian $H = -\Delta$ inside a convex domain of dimension $d \geq 2$ in $\mathbb{R}^d$:

[Ivanovici-Lebeau-Planchon, 14]

$$\|\cos(s\sqrt{H})\psi_\ell(r^2 H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \left(\frac{r}{s}\right)^{\frac{d-1}{2}} \left(\frac{r}{|L-s|}\right)^{\frac{d+1}{2}+\frac{1}{4}}.$$

In specific situations, complex phenomena seem to appear near the boundary of the light cone...

Weak assumption on the wave propagator

For all $s \in (0,1)$, for all $r > 0$ and any two balls B, $\widetilde{B}$ of radius r

$$\|\cos(s\sqrt{H})\psi_\ell(r^2H)\|_{L^2(B)\to L^2(\widetilde{B})} \lesssim \left(\frac{r}{r+s}\right)^{\frac{d-1}{2}}.$$

No behaviour near the boundary of the light cone is assumed.

Theorem

Assume $d > 1$, $m \geq \lceil \frac{d}{2} \rceil$, and the previous weak assumption is satisfied, then for all balls B, $\widetilde{B}$ of radius r and all $\varepsilon > 0$:

$$e^{itH}\psi_{m'}(h^2 H) \text{ satisfies } (H_m(t^{-\frac{d-2}{2}} h^{-2}))$$

for $h^2 \leq t \leq h^{1+\varepsilon}$ and $m' \geq 0$.

Summary

Weak dispersion for the wave propagator $\Rightarrow$ weak dispersion for the Schrödinger propagator.

Theorem

Assume $d > 1$, $m \geq \lceil \frac{d}{2} \rceil$, and the previous weak assumption is satisfied, then for every $h^2 \leq t \leq h^{1+\varepsilon}$ and all $2 \leq p \leq +\infty$ and $2 \leq q < +\infty$ satisfying

$$\frac{2}{p} + \frac{d-2}{q} = \frac{d-2}{2},$$

every solution $u(t,.) = e^{itH}u_0$ of the problem

$$\begin{cases} i\partial_t u + Hu = 0 \\ u_{|t=0} = u_0 \end{cases},$$

satisfies local-in-time Strichartz estimates with loss of derivatives

$$\|u\|_{L^p([-1,1],L^q)} \lesssim \|u_0\|_{W^{\frac{1+\varepsilon}{p}+2(1-\frac{2}{q}),2}}.$$

Remark on the loss of derivatives:
The loss is interesting when

$$\frac{1+\varepsilon}{p} + 2\left(1-\frac{2}{q}\right) \leq \frac{2}{p}.$$

Moreover

$$\frac{2}{p} + \frac{d-2}{q} = \frac{d-2}{2}.$$

Hence

$$d \geq \frac{8}{1-\varepsilon} + 2.$$

If $d > 10$, one can find such an $\varepsilon \in (0,1)$.

One thing to remember:

$L^2(B) - L^2(\widetilde{B})$ dispersive estimates for the wave propagator

$$\Downarrow$$

H^1 – BMO dispersive estimates for the Schrödinger operator

$$\Downarrow$$

$L^p L^q$ Strichartz inequalities for the Schrödinger operator

- A good understanding of the wave propagator in various settings will help to detect whereas the method can apply:
  - The proof of (DG) $\Leftrightarrow$ (4) may allow us to show that gaussian upper bounds (UE) imply a dispersion for $\cos(s\sqrt{H})$;
  - Klainerman's commuting vector fields method may give a suitable $L^1 - L^\infty$ dispersive estimates for $\cos(s\sqrt{H})$ in various settings (mild assumption on the geometry of X, or $H = -\mathrm{div}(A\nabla)$ with no/minimal regularity on A);
- Find new examples where we can apply our method to derive Strichartz estimates in general settings;
- Perturbation of H with a potential V with no regularity.

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Thank you for your attention !