Propagations de fronts gouvernées par des lois non-locales

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Outline

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Geometric equations

We are interested in geometric equations governing the movement of a family $K = \{K(t)\}_{t\in[0,T]}$ of compact subsets of $\mathbb{R}^N$:

$$V_{x,t} = f(x, t, \nu_{x,t}, A_{x,t}, K). \tag{1}$$

- $V_{x,t}$ is the normal velocity of a point x of $\partial K(t)$.
- $\nu_{x,t}$ is the unit exterior normal to $K(t)$ at $x \in \partial K(t)$.
- $A_{x,t} = [-\frac{\partial \nu_i}{\partial x_j}(x,t)]$ is the curvature matrix of $K(t)$ at $x \in \partial K(t)$.
- $K \mapsto f(x, t, \nu_{x,t}, D\nu, K)$ is a **non-local dependence** in the whole front K (up to time t).

Level-set method

This method was developed by Sethian and Osher in 1988.

Assume that the front $\Gamma(t) = \partial K(t)$ is smooth, and that there exists a smooth $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ such that

$$K(t) = \{x \in \mathbb{R}^N;\ u(x,t) > 0\}, \quad \Gamma(t) = \{x \in \mathbb{R}^N;\ u(x,t) = 0\},$$

and $Du(x,t) \neq 0$ when $x \in \Gamma(t)$.

Our law is $$V_{x,t} = f(x, t, \nu_{x,t}, A_{x,t}, K).$$

But

$$V_{x,t} = \frac{u_t(x,t)}{|Du(x,t)|}, \quad \nu_{x,t} = -\frac{Du(x,t)}{|Du(x,t)|},$$

$$A_{x,t} = \frac{1}{|Du(x,t)|}\left(I - \frac{Du(x,t)Du(x,t)^T}{|Du(x,t)|^2}\right) D^2u(x,t),$$

so that u satisfies the **level-set** equation associated to (1) :

$$u_t(x,t) = f\left(x, t, -\frac{Du}{|Du|}, \frac{1}{|Du|}\left(I - \frac{Du\, Du^T}{|Du|^2}\right) D^2u, \{u \geq 0\}\right) |Du(x,t)|$$

$$= F(x, t, Du, D^2u, \mathbf{1}_{\{u \geq 0\}})|Du(x,t)|.$$

Level-set method

To generalize the preceding evolution to non-smooth fronts, we realize the following program :

1. Find $u_0 : \mathbb{R}^N \to \mathbb{R}$ such that
$$K_0 = \{u_0 \geq 0\}, \quad \Gamma_0 = \{u_0 = 0\}.$$
2. Solve in the viscosity sense the problem
$$\begin{cases} u_t(x,t) = F(x,t,Du,D^2u,\mathbf{1}_{\{u\geq 0\}})|Du(x,t)| & \text{in } \mathbb{R}^N \times (0,T) \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{2}$$
3. Set $K(t) = \{x \in \mathbb{R}^N;\ u(x,t) \geq 0\}, \quad \Gamma(t) = \{x \in \mathbb{R}^N;\ u(x,t) = 0\}.$

Examples of local laws

- The eikonal equation $V_{x,t} = c(x,t)$ of level-set equation

$$u_t = c(x,t)|Du|.$$

- The motion by mean curvature equation $V_{x,t} = H_{x,t} = Tr(A_{x,t})$ of level-set equation

$$u_t = \mathrm{div}\left(\frac{Du}{|Du|}\right)|Du| = \Delta u - \frac{\langle D^2uDu, Du\rangle}{|Du|^2}.$$

Main issue

Main problem : f is not necessarily monotone in K :

$$K \subset K' \text{ does not imply}$$

$$f(x,t,\nu,A,K) \leq f(x,t,\nu,A,K').$$

$\Rightarrow$ No inclusion principle :

$$K_0 \subset K_0' \text{ does not imply } K(t) \subset K'(t) \text{ for all } t \geq 0.$$

$\Rightarrow$ The classical techniques for building a viscosity solution fail.

Example : dislocation dynamics

Recently, the dislocation dynamics model,

$$V_{x,t} = c_0(\cdot, t) \star \mathbf{1}_{K(t)}(x) + c_1(x, t)$$

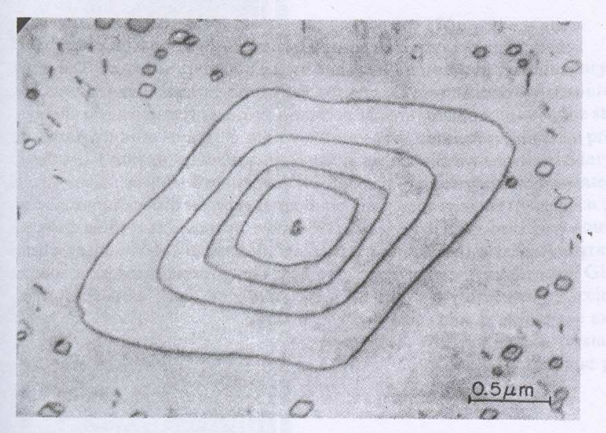
of associated level-set equation

$$u_t(x, t) = \left[c_0(\cdot, t) \star \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x, t)\right] |Du(x, t)|,$$

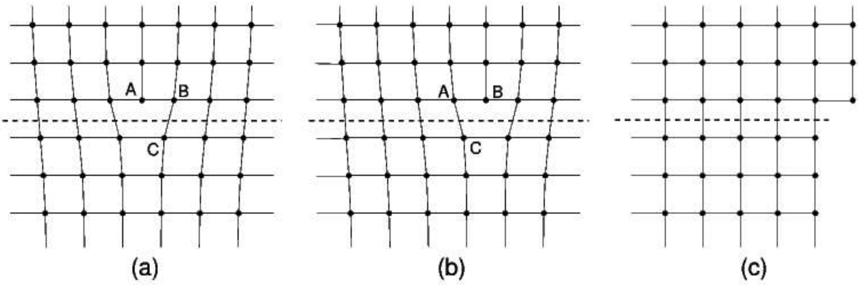
has drawn a lot of attention.

- $c_0(\cdot, t) \star \mathbf{1}_{K(t)}(x) = \int_{K(t)} c_0(x - y, t)\, dy$ is a **nonlocal driving force.**
- c_1 is a prescribed driving force.

Dislocations



Movement of a dislocation



Known results

- Alvarez, Hoch, Le Bouar, Monneau :

  Short time existence and uniqueness of a viscosity solution, for a smooth initial data.

- Alvarez, Cardaliaguet, Monneau / Barles, Ley :

  Long time existence and uniqueness of a viscosity solution, for an initial shape K_0 having the **interior ball property**, under the condition that

$$c_1(x,t) \geq \|c_0(\cdot,t)\|_1 \quad \forall (x,t) \in \mathbb{R}^N \times [0,+\infty).$$

Key point : Under the assumption that

$$c_1(x,t) \geq \|c_0(\cdot,t)\|_1 \quad \forall (x,t) \in \mathbb{R}^N \times [0,+\infty),$$

the motion, although non-local, is **non-decreasing** :

the velocity f of the front is non-negative, because

$$\forall K \subset \mathbb{R}^N, \quad c_0(\cdot,t) \star \mathbf{1}_K(x) + c_1(x,t) \geq 0.$$

What remains of this if no assumption is made :

- On the monotonicity of f,
- On its sign ?

Outline

Motivation of the definition of weak solutions

Let us investigate the stability of solutions : suppose that u_ε are solutions to

$$\begin{cases} (u_\varepsilon)_t(x,t) = \left[c_0(\cdot,t) \star \mathbf{1}_{\{u_\varepsilon(\cdot,t)\geq 0\}}(x) + c_1(x,t)\right] |Du_\varepsilon(x,t)|, \\ u_\varepsilon(x,0) = u_0^\varepsilon(x), \end{cases}$$

with $u_0^\varepsilon \to u_0$ uniformly.

Then standard estimates imply that $u_\varepsilon \to u$ in $C^0(\mathbb{R}^N \times [0,T])$, and that $\mathbf{1}_{\{u_\varepsilon \geq 0\}} \rightharpoonup \chi$ weakly-$\star$ in $L^\infty_{loc}(\mathbb{R}^N \times [0,T],[0,1])$.

From Barles' stability result, we have in the L^1 viscosity sense :

$$\begin{cases} u_t(x,t) = [c_0(\cdot,t) \star \chi(\cdot,t)(x) + c_1(x,t)]\,|Du(x,t)|, \\ u(x,0) = u_0(x), \end{cases}$$

and

$$\mathbf{1}_{\{u>0\}} \leq \liminf \mathbf{1}_{\{u_\varepsilon \geq 0\}} \leq \chi \leq \limsup \mathbf{1}_{\{u_\varepsilon \geq 0\}} \leq \mathbf{1}_{\{u\geq 0\}}$$

Definition of weak solutions

Let $u : \mathbb{R}^N \times [0, T]$ be a continuous function. We say that u is a weak solution of (2) if there exists $\chi \in L^\infty(\mathbb{R}^N \times [0, T], [0, 1])$ such that :

1. u is the L^1 viscosity solution of

$$\begin{cases} u_t(x,t) = F(x,t,Du,D^2u,\chi)|Du(x,t)| & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

2. For almost all $t \in [0, T]$,

$$\mathbf{1}_{\{u(\cdot,t)>0\}} \leq \chi(\cdot, t) \leq \mathbf{1}_{\{u(\cdot,t)\geq 0\}}.$$

Moreover, we say that u is a classical solution of (2) if in addition, for almost all $t \in [0, T]$ and almost everywhere in $\mathbb{R}^N$,

$$\{u(\cdot, t) > 0\} = \{u(\cdot, t) \geq 0\}.$$

References

Similar definitions (with existence results) can be found in :

- Giga, Goto, Ishii (SIAM J. Math. Anal., 1992)
- Soravia, Souganidis (SIAM J. Math. Anal., 1996)
- Hilhorst, Logak, Schätzle (Interfaces Free Bound., 2000) : phase-field approach for the evolution law : $V_{x,t} = H_{x,t} - Vol(K)$.

Existence theorem : assumptions on F

(H1) For all $\chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$, the problem

$$\begin{cases} u_t(x,t) = F(x,t,Du,D^2u,\chi)|Du(x,t)| & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

- has a continuous L^1 viscosity solution u with $|u| \leq M$.
- satisfies a comparison principle.

(H2) If $\chi_n \rightharpoonup \chi$ weakly-$\star$ in $L^\infty(\mathbb{R}^N \times [0,T],[0,1])$, then

$$\int_0^t F(x,s,p,X,\chi_n)\,ds \to \int_0^t F(x,s,p,X,\chi)\,ds$$

locally uniformly for $t \in [0,T]$ as $n \to +\infty$.

Existence theorem

Theorem (Barles, Cardaliaguet, Ley, M.)

Under assumptions $(H1)$ *and* $(H2)$, *the problem*

$$\begin{cases} u_t(x,t) = F(x,t,Du,D^2u,\mathbb{1}_{\{u\geq 0\}})|Du(x,t)| & in\,\mathbb{R}^N \times (0,T) \\ u(0,x) = u_0(x) & in\,\mathbb{R}^N. \end{cases}$$

has at least a weak solution.

Idea of proof

Let us consider the set-valued mapping

$$\xi: \ \chi \in L^\infty(\mathbb{R}^N \times [0,T],[0,1])$$

$$\mapsto u \text{ viscosity solution of } \begin{cases} u_t(x,t) = F(x,t,Du,D^2u,\chi)|Du(x,t)| \\ u(0,x) = u_0(x) \end{cases}$$

$$\mapsto \{\chi';\ \mathbf{1}_{\{u(\cdot,t)>0\}} \leq \chi'(\cdot,t) \leq \mathbf{1}_{\{u(\cdot,t)\geq 0\}} \text{ for almost all } t \in [0,T]\}.$$

Clearly, there exists a weak solution to (2) if and only if there exists a fixed point of χ of ξ in the sense that $\chi \in \xi(\chi)$.

In this case the corresponding u is a weak solution to (2).

We prove existence of a fixed point of ξ by Kakutani's fixed point theorem.

Verification of the assumptions

- The map ξ is well defined thanks to (H1).
- For any $\chi \in L^\infty(\mathbb{R}^N \times [0,T], [0,1])$, $\xi(\chi)$ is convex and compact for the L^∞-weak-$\star$ topology (closed and bounded).
- The map ξ is upper semicontinuous for this topology, in the sense that if

$$\chi_n \underset{L^\infty-weak-\star}{\rightharpoonup} \chi \quad \text{and} \quad \chi'_n \in \xi(\chi_n) \underset{L^\infty-weak-\star}{\rightharpoonup} \chi',$$

then

$$\chi' \in \xi(\chi).$$

Discussion of the definition of weak solutions

Main advantages of the definition :

- Existence result under mild conditions on the dynamics.
- Stability results.

Main drawback :

The fattening phenomenon plays a central role :

- We can not identify χ on the set $\{u = 0\}$.
- Strange behavior of $t \mapsto \chi(\cdot, t)$.

Main remaining difficulty :

- Uniqueness is almost completely open.

Outline

The equation

This is a joint work with N. Forcadel.

We consider the example of dislocation dynamics with a mean curvature term :

$$V_{x,t} = H_{x,t} + c_0(\cdot, t) \star \mathbf{1}_{K(t)}(x) + c_1(x, t). \qquad (3)$$

where $H_{x,t} = Tr(A_{x,t})$ is the mean curvature of $\partial K(t)$ at a point x.

Level-set equation

The level-set equation corresponding to (3) is

$$u_t(x,t) = \left[\operatorname{div}\left(\frac{Du}{|Du|}\right)(x,t) + c_0(\cdot,t) \star \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x) + c_1(x,t)\right] |Du(x,t)|. \tag{4}$$

Abscence of sign of c_0 or comparison between $c_0,\ c_1$

$\Rightarrow$ Non-monotone problem.

Known results

Forcadel :

Short time existence and uniqueness of a viscosity solution, provided the initial shape is a graph or a Lipschitz curve.

Main issues

1. Can we provide weak solutions to (4) ?

2. Does the mean curvature term have a regularizing effect ?

3. If the initial shape is smooth enough, is there a unique smooth evolution for small times ?

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   Yes, thanks to our existence theorem.

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Main issues

1. Can we provide weak solutions to (4) ?

   Yes, thanks to our existence theorem.

2. Does the mean curvature term have a regularizing effect ?

   We search for $\chi(\cdot, t)$ in the particular form $\mathbf{1}_{E(t)}$ with some regularity in time.

3. If the initial shape is smooth enough, is there a unique smooth evolution for small times ?

Minimizing movements

We build E as a minimizing movement for our evolution law : following Almgren, Taylor and Wang (1993), we discretize the equation

$$V_{x,t} = H_{x,t} + c_0(\cdot, t) \star \mathbf{1}_{K(t)}(x) + c_1(x, t)$$

in time. Let h be a time step.

We are going to construct a sequence of sets $E_h(k)$, for $k \in \mathbb{N}$ such that $kh \leq T$, whose evolution with k is a discretization of (3).

Approximation of the velocity

Assume that this sequence is built.

Then, for $x \in \partial E_h(k+1)$ with $x \notin E_h(k)$,

$$\frac{d_{E_h(k)}(x)}{h}$$

is an approximation of the velocity of x at time $t = (k+1)h$.

Likewise, if $x \in E_h(k)$,

$$-\frac{d_{E_h(k)}(x)}{h}$$

is an approximation of the velocity of x at time $t = (k+1)h$.

Discretization

We therefore wish to construct a sequence of sets $E_h(k)$ such that for all $x \in \partial E_h(k+1)$,

$$\pm \frac{d_{E_h(k)}(x)}{h} = H_{x,(k+1)h} + c_0(\cdot, (k+1)h) \star \mathbf{1}_{E_h(k+1)}(x) + c_1(x, (k+1)h), \tag{5}$$

where we take the $+$ sign if $x \notin E_h(k)$, the $-$ sign otherwise.

This corresponds to an **implicit time discretization** of (3).

Corresponding gradient flow

We construct $E_h(k+1)$ by seeing the equation

$$\pm\frac{d_{E_h(k)}(x)}{h} = H_{x,(k+1)h} + c_0(\cdot,(k+1)h) \star \mathbf{1}_{E_h(k+1)}(x) + c_1(x,(k+1)h)$$

as the Euler equation corresponding to the minimization of the functionnal

$$E \mapsto \mathcal{F}(h, k+1, E,\ E_h(k))$$

$$\begin{aligned} &= P(E) + \frac{1}{h}\int_{E\Delta E_h(k)} d_{\partial E_h(k)}(x)\,dx \\ &- \int_E \left(\frac{1}{2}c_0(\cdot,(k+1)h) \star \mathbf{1}_E(x) + c_1(x,(k+1)h)\right)\,dx. \end{aligned}$$

Definition (Minimizing movement)

Let $E_0 \in \mathcal{P}$. We say that $E : [0, T] \to \mathcal{P}$ is a minimizing movement associated to $\mathcal{F}$ with initial condition E_0 if there exist $h_n \to 0^+$ and sets $E_{h_n}(k) \in \mathcal{P}$ for all $k \in \mathbb{N}$ verifying $kh_n \leq T$, such that :

1. $E_{h_n}(0) = E_0$.
2. For any $n, k \in \mathbb{N}$ with $(k+1)h_n \leq T$,

   $E_{h_n}(k+1)$ minimizes the functional $E \to \mathcal{F}(h_n, k+1, E, E_{h_n}(k))$.
3. For any $t \in [0, T]$, $E_{h_n}([t/h_n]) \to E(t)$ in $L^1(\mathbb{R}^N)$ as $n \to +\infty$.

Results

Under adapted regularity assumptions on c_0 and c_1, we obtained :

Theorem (Forcadel, M.)

Let $E_0 \in \mathcal{P}$ with $\mathcal{L}^N(\partial E_0) = 0$. Then :

1. *There exist **Hölder continuous** minimizing movements associated to $\mathcal{F}$ with initial condition E_0.*
2. *The corresponding u is a weak solution of (4).*
3. *If E_0 is a compact domain with uniformly $C^{3+\alpha}$ boundary, there exists a small time $t_0 > 0$ and a smooth evolution $\{E_r(t)\}_{0 \leq t \leq t_0}$ with $C^{3+\alpha}$ boundary, starting from E_0, with velocity given by (3).*

   Moreover, any minimizing movement E associated to $\mathcal{F}$ with initial condition E_0 verifies $E(t) = E_r(t)$ for all $t \in [0, t_0]$ and almost everywhere in $\mathbb{R}^N$.

The main ingredients of proof are :

- A lower density bound for $\mathcal{F}$-minimizers : $P(E, B_\rho(x)) \geq \beta\rho^{N-1}$.
- A Distance-Volume comparison to estimate $|E_h(k+1)\Delta E_h(k)|$.
- A regularity result for $\mathcal{F}$-minimizers, so that the Euler-Lagrange equation corresponding to our minimizing procedure is the discretized equation.
- Sub/super pairs of solutions of Cardaliaguet and Pasquignon.

Outline

The system

This is a joint work with G. Barles, O. Ley and P. Cardaliaguet.

We are now interested in the asymptotic behavior as $\varepsilon \to 0$ of the following system,

$$\begin{cases} u_t^\varepsilon - \varepsilon \Delta u^\varepsilon = \dfrac{1}{\varepsilon} f(u^\varepsilon, v^\varepsilon), \\ v_t^\varepsilon - \Delta v^\varepsilon = g(u^\varepsilon, v^\varepsilon), \end{cases} \tag{6}$$

in $\mathbb{R}^N \times (0, T)$ ($N \geq 3$), where

$$\begin{cases} f(u, v) = u(1-u)(u-a) - v \quad (0 < a < 1), \\ g(u, v) = u - \gamma v \qquad (\gamma > 0). \end{cases}$$

The initial conditions are $v(\cdot, 0) = 0$ and $u(\cdot, 0) = u_0$.

The asymptotic system

We expect a limit system of model case

$$\begin{cases} u_t(x,t) = c(v(x,t))|Du(x,t)|, \\ v_t(x,t) - \Delta v(x,t) = \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x), \end{cases} \tag{7}$$

for $(x,t) \in \mathbb{R}^N \times (0,T)$.

The function c is Lipschitz continuous on $\mathbb{R}$.

The Heat equation part

For $\chi \in L^\infty(\mathbb{R}^N \times (0, T))$, the solution of

$$\begin{cases} v_t(x,t) - \Delta v(x,t) = \chi(x,t) \\ v(x,0) = 0 \end{cases}$$

is explicitly given by the formula

$$v(x,t) = \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)\chi(y,s)\, dyds,$$

where G is the Green function defined by

$$G(y,s) = \frac{1}{(4\pi s)^{N/2}} e^{-\frac{|y|^2}{4s}}.$$

The Heat equation part

Hence problem (7) reduces to the equation

$$u_t(x,t) = c\left(\int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\mathbf{1}_{\{u(\cdot,s)\geq 0\}}(y)\,dyds\right)|Du(x,t)| \tag{8}$$

$$= F(x,t,\mathbf{1}_{\{u\geq 0\}})|Du(x,t)|.$$

with

$$F(x,t,\chi) = c\left(\int_0^t \int_{\mathbb{R}^N} G(x-y,t-s)\chi(y,s)\,dyds\right).$$

Abscence of sign or monotonicity of $c \Rightarrow$ Non-monotone problem.

Main issues

1. Can we provide weak solutions to (8) ?

2. If $c > 0$, is this solution unique ?

Main issues

1. Can we provide weak solutions to (8) ?

Theorem (Giga, Goto, Ishii '92 / Soravia, Souganidis '96)

There exist weak solutions to (8).

If $c \geq 0$*, these solutions are classical.*

2. If $c > 0$, is this solution unique ?

Additionnal assumptions :

- The initial set $K_0 \subset B(0, R)$ is the closure of a bounded open subset of $\mathbb{R}^N$ with C^2 boundary. (Technical assumption to be relaxed)
- There exist $\delta > 0$ and $L > 0$ such that $\delta \leq c(x) \leq L$ in $\mathbb{R}$.

Let u_1 and u_2 be two classical solutions of

$$\begin{cases} u_t(x,t) = c(v(x,t))|Du(x,t)|, \\ v_t(x,t) - \Delta v(x,t) = \mathbf{1}_{\{u(\cdot,t)\geq 0\}}(x). \end{cases}$$

with initial conditions $v(\cdot,0) = 0$ and $u(\cdot,0) = u_0$.

Let us set, for $i = 1, 2$ and $t \in [0, T]$,

$$K_1(t) = \{u_1(\cdot, t) \geq 0\},\ K_2(t) = \{u_2(\cdot, t) \geq 0\},$$

and

$$v_i : (x, t) \mapsto \int_0^t \int_{\mathbb{R}^N} G(x - y, t - s) \mathbf{1}_{K_i(s)}(y)\, dyds$$

the solution of

$$\begin{cases} (v_i)_t - \Delta v_i = \mathbf{1}_{K_i} & \text{in } \mathbb{R}^N \times (0, T), \\ v_i(\cdot, 0) = 0 & \text{in } \mathbb{R}^N. \end{cases}$$

It suffices to prove that $K_1 = K_2$, since this implies that $v_1 = v_2$, and finally $u_1 = u_2$.

We estimate for any $t \in [0, T]$,

$$d_{\mathcal{H}}(\{u_1(\cdot,t) \geq 0\}, \{u_2(\cdot,t) \geq 0\})$$

$$\leq T\,k(N,T)\,\|c(v_1) - c(v_2)\|_{L^\infty(\mathbb{R}^N \times [0,T])}$$

$$\leq T\,k(N,T)\,\|c'\|_\infty \|v_1 - v_2\|_{L^\infty(\mathbb{R}^N \times [0,T])}.$$

For any $x \in \mathbb{R}^N$ and $t \in [0, T]$,

$$|v_1(x,t) - v_2(x,t)|$$

$$\leq \left| \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)(\mathbf{1}_{K_1(s)}(y) - \mathbf{1}_{K_2(s)}(y))\, dyds \right|$$

$$\leq \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)(\mathbf{1}_{K_1(s)\setminus K_2(s)}(y) + \mathbf{1}_{K_2(s)\setminus K_1(s)}(y))\, dyds.$$

Set $\quad r = \sup_{t\in[0,T]} d_{\mathcal{H}}(K_1(t), K_2(t)) \leq LT.$

Then if $B = \bar{B}(0,1)$, we have

$$|v_1(x,t) - v_2(x,t)|$$

$$\leq \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)(\mathbf{1}_{(K_1(s) \quad)\setminus K_2(s)}(y) + \mathbf{1}_{(K_2(s) \quad)\setminus K_1(s)}(y))\, dyds.$$

Set $\quad r = \sup_{t\in[0,T]} d_{\mathcal{H}}(K_1(t), K_2(t)) \leq LT.$

Then if $B = \bar{B}(0,1)$, we have

$$|v_1(x,t) - v_2(x,t)|$$

$$\leq \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)(\mathbf{1}_{(K_2(s)+rB)\setminus K_2(s)}(y) + \mathbf{1}_{(K_1(s)+rB)\setminus K_1(s)}(y))\, dyds.$$

The key is to provide the estimate

$$\int_0^t \int_{\mathbb{R}^N} G(x-y, t-s) \mathbf{1}_{(K_i(s)+rB)\setminus K_i(s)}(y)\, dyds \leq C\, r,$$

so that for any $t \in [0, T]$,

$$d_{\mathcal{H}}(\{u_1(\cdot, t) \geq 0\}, \{u_2(\cdot, t) \geq 0\}) \leq T\, k(N, T)\, \|c'\|_\infty\, 2C\, r,$$

and we would obtain that $u_1 = u_2$ on $\mathbb{R}^N \times [0, T]$ for T small enough.

Interior cone property

However, the estimation

$$\int_0^t \int_{\mathbb{R}^N} G(x-y, t-s)\mathbf{1}_{(K(s)+rB)\setminus K(s)}(y)\, dyds \leq C\, r,$$

does not hold for any K :

1. it requires at least that $(K(s)+rB)\setminus K(s)$ be small in $L^1(\mathbb{R}^N)$...
2. ... which is not automatic since

$$Vol((K(s)+rB)\setminus K(s)) \approx Per(K(s))\, r \ldots$$

3. ... and would not be enough since $\chi \mapsto v$ solution of $v_t - \Delta v = \chi$ is not continuous from L^1 to L^∞.

$\rightarrow$ We need certain regularity for the sets $K_i = \{u_i \geq 0\}$.

This regularity is the **interior cone property** :

Definition

Let K be a compact subset of $\mathbb{R}^N$. We say that K has the interior cone property of parameters ρ and θ if $0 < \rho < \theta$ and

$$\forall\, x \in \partial K,\ \exists\, \nu \in \mathbb{S}^{N-1} \text{ such that } C_{\nu,x}^{\rho,\theta} := x + [0,\theta]\bar{B}_N(\nu, \rho/\theta) \subset K,$$

where $\bar{B}_j(x,r)$ is the closed ball of $\mathbb{R}^j$ of radius r centered at x.

To prove our uniqueness result, we therefore need three ingredients :

1. The propagation of the interior cone property for solutions of the eikonal equation :

   $K_1(t) = \{u_1(\cdot, t) \geq 0\}$ and $K_2(t) = \{u_2(\cdot, t) \geq 0\}$ have the interior cone property for all $t \in [0, T]$, for some parameters ρ and θ independent of t.

2. A perimeter estimate for sets having the interior cone property.

3. An estimate on the L^∞ norm of the solutions of the r-perturbed equation
$$\begin{cases} v_t(x,t) - \Delta v(x,t) = \mathbf{1}_{(K(t)+rB)\setminus K(t)}(x) \\ v(\cdot, 0) = 0. \end{cases}$$
   in function of r for such a K.

Propagation of the interior cone property

Theorem

Let K_0 be the closure of a bounded open subset of $\mathbb{R}^N$ with C^2 boundary, and let $c : \mathbb{R}^N \times [0,T] \to \mathbb{R}^N$ satisfy the following assumptions : there exist $\delta, L, M > 0$ such that :

$$\begin{cases} \delta \leq c \leq L, \\ c \text{ is continuous on } \mathbb{R}^N \times [0,T], \\ \forall\, t \in [0,T],\ \ c(\cdot,t) \text{ is differentiable in } \mathbb{R}^N \text{ with } \|Dc\|_\infty \leq M. \end{cases}$$

Let u be the unique uniformly continuous viscosity solution of

$$\begin{cases} u_t(x,t) = c(x,t)|Du(x,t)| & \text{in } \mathbb{R}^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

Then there exist $\rho > 0$ and $\theta > 0$ depending only on c and K_0 such that

$$K(t) = \{x \in \mathbb{R}^N;\ u(x,t) \geq 0\}$$

has the interior cone property of parameters ρ and θ for all $t \in [0, T]$.

Sets with the interior cone property

Theorem

Let K be a compact subset of $\mathbb{R}^N$ having the cone property of parameters ρ and θ.

Then there exists a positive constant $C_0 = C_0(N, \rho, \theta/\rho)$ such that for all $R > 0$,

$$\mathcal{H}^{N-1}(\partial K \cap \bar{B}(0,R)) \leq C_0\, \mathcal{L}^N(K \cap \bar{B}(0, R + \rho/4)).$$

The r-perturbed equation

Theorem

Let $\{K(t)\}_{t\in[0,T]} \subset \bar{B}_N(0,D)\times[0,T]$ *be a bounded family of compact subsets of* $\mathbb{R}^N$ *having the interior cone property of parameters* ρ *and* θ *with* $0<\rho<\theta<1$*, and let us set, for any* $x\in\mathbb{R}^N$*,* $t\in[0,T]$ *and* $r\geq 0$*,*

$$\phi(x,t,r)=\int_0^t\int_{\mathbb{R}^N} G(x-y,t-s)\,\mathbf{1}_{(K(s)+rB)\setminus K(s)}(y)\,dyds.$$

Then for any $r_0>0$*, there exists a constant* $C_1=C_1(T,N,D,r_0,\rho,\theta/\rho)$ *such that for any* $x\in\mathbb{R}^N$*,* $t\in[0,T]$ *and* $r\in[0,r_0]$*,*

$$|\phi(x,t,r)|\leq C_1\,r.$$

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