Multivariate subdivision and a simple formula

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The curse of dimension

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  - **Task:** Determine properties of φ from a.

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Building blocks of refinement equation:

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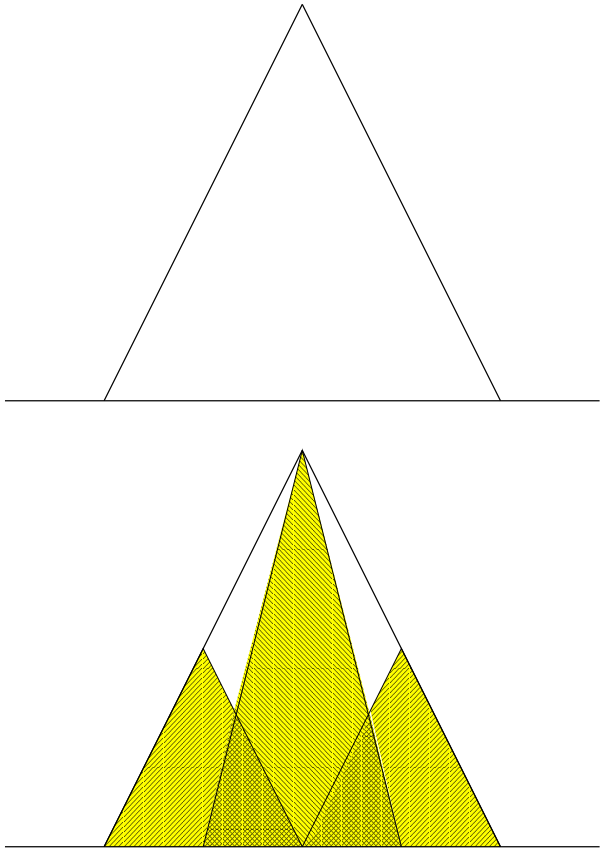

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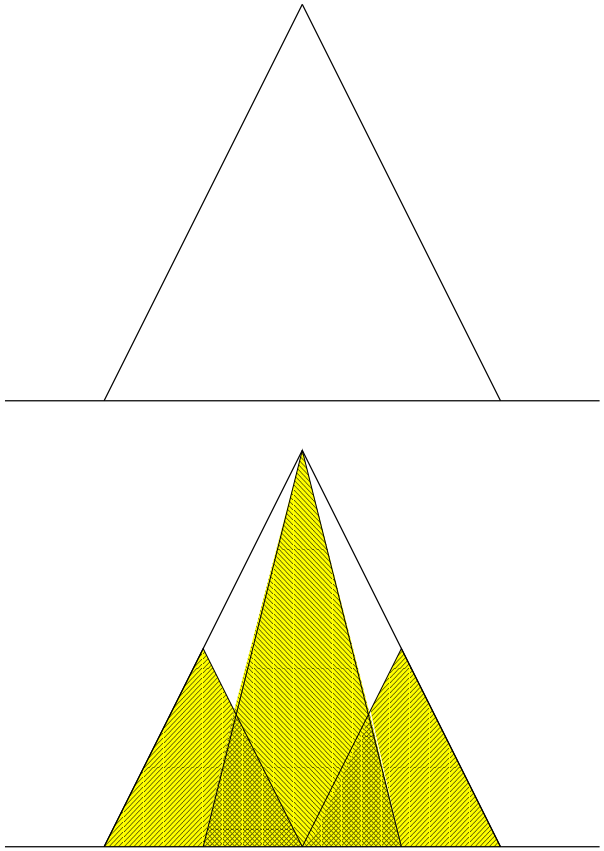

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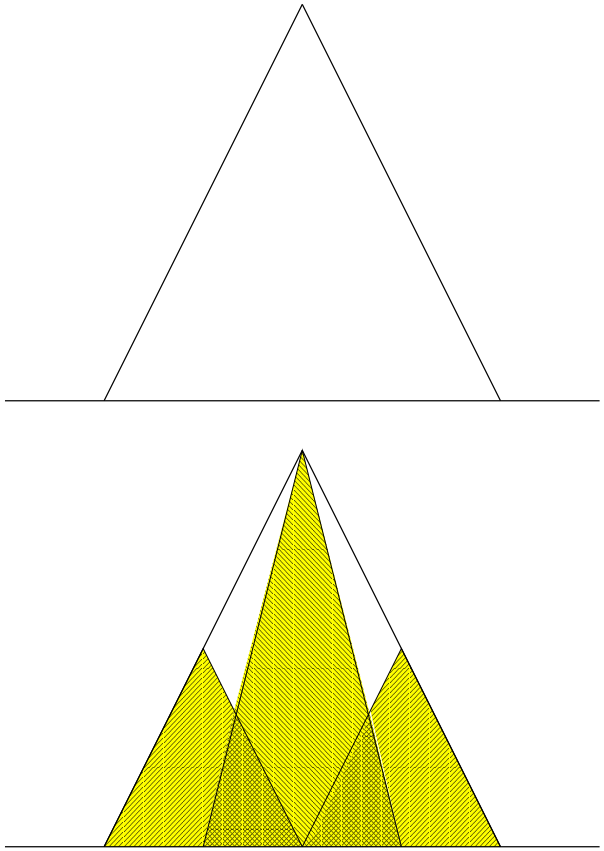
- In the example: $a = \left(\ldots, 0, \frac{1}{2}, 1, \frac{1}{2}, 0, \ldots\right)$.



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Remark. *Norm* $\leftrightarrow$ *spaces from which* c *and* f_c *are taken.*

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Example. $(S_a\, c)^* = a^*(z)\; c^*\left(z^2\right)$.

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"Factorization" by means of difference operators.

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Quotient ideal & representation matrix

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Defines and computes representation matrix $\mathbf{B}^*$.

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