SIMULATION OF MUFFLER'S 
BY A HOMOGENIZED FINITE ELEMENT METHOD

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In this work, we are interested in the modelling of the acoustic attenuation of exhaust mufflers including perforated ducts, and its numerical computation. The study is worked out in harmonic time regime, for the two-dimensional case. The hole diameter and the center-to-center distance between consecutive holes are supposed of same order, and small compared to the size of the muffler. The formulation is derived by using multiscale techniques and matching the asymptotic expansions. The numerical method couples finite elements in the muffler with modal decomposition in the inlet and the outlet of the duct.

1. Introduction

In order to simplify the study of propagation of waves inside an exhaust silencer, some assumptions are generally used to reduce the problem to the one-dimensional case. Consequently, three-dimensional effects are neglected (cf. Munjal et al\textsuperscript{1}, Lienard\textsuperscript{2}). However some studies use the finite element method for the numerical computation (cf. Wang\textsuperscript{3}, Ross\textsuperscript{4}, Tanaka et al\textsuperscript{5}). The more difficult point is to generate a mesh that takes into account the different scales involved in the design of such a system: small scale for the holes, medium scale for the inner tube and large scale for the cavity. In this work we propose a numerical method that couples finite elements in the muffler with modal decomposition in the inlet and the outlet of the duct. The effects of the perforated ducts are taken into account by

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means of an impedance condition, derived by using asymptotic multi-scale techniques (see Sanchez et al\textsuperscript{6}, Bergman et al\textsuperscript{7} and Nguetseng\textsuperscript{8} who carried out a similar study for the Laplace operator).

The perforated ducts used in mufflers design are of finite length, and periodic in the propagation direction. We shall neglect the ducts extremities effects and use homogenization method.

We suppose that the center-to-center distance between two consecutive holes is equal to $\varepsilon$. The proposed method is based on the solution's asymptotic expansion with respect to the parameter $\varepsilon$.

It consists in splitting the computation domain representing the muffler into two regions: the first one is close to the perforated area, and the second one is the external domain to the boundary layer. In each region the proper asymptotic expansion of the solution is derived using a multi-scale method and taking into account the geometric specifications of the area. In the following we will call inner expansion the one defined in the boundary layer, and outer expansion the one defined in the second region.

As the determination of the transmission losses requires only the computation of the solution $u_{e}$ at the inlet and the outlet ducts of the muffler, we are mainly interested in the outer expansion of $u_{e}$. An approximation of $u_{e}$ is therefore determined, with the computation of the two first terms of the outer expansion. The evaluation of these two terms requires to solve boundary value problems set in a simpler geometry, where the perforated area has been removed and replaced by transmission conditions. These conditions are obtained after matching both expansions. It will be shown, that the problems satisfied by the terms of the inner expansion do not have to be completely evaluated. Only their behavior at infinity has to be determined. The matching rules of the inner and the outer expansion are those used by Nguetseng\textsuperscript{8} for the Laplace equation. We apply them to the Helmholtz operator.

Let us point out that the homogenization method allowed us to derive a problem, satisfied by the first order approximation, which may be easily solved by a finite element method. In particular, the mesh has not to take into account the perforations.

In this work we will consider the two-dimensional case. The extension to the three-dimensional case should not add specific difficulties. The outline of the article appears as follows: in section 2 we give a description of the geometry and the problem which we suggest to solve. Section 3 is dedicated to the definition of the asymptotic expansions in both zones of the silencer. Sections 4 and 5 define completely the problems satisfied by every term of the outer expansion. Numerical applications are described in section 6. Some theoretical and technical results are left in appendix.

2. Description of the geometry
The muffler is represented by a rectangular box, in the $x_1x_2-$ plane, crossed by a tube. Let $L$ be the length of the box. The tube is periodically perforated inside the box. For sake of simplicity, the tube is assumed to be infinite, which allows to neglect the effect of the exterior medium. By symmetry considerations, we can restrict the problem to a half part
of the muffler. Moreover, in order to use a finite element method, we bound the domain by two fictitious boundaries located in the tube, on each side of the muffler. Finally, the muffler is represented by $\Omega_{\varepsilon}$, a bounded open domain of $\mathbb{R}^2$, as shown in figure 1. The perforated wall $\Sigma_{\varepsilon}$, located at $x_2 = 0$, is composed of small segments, with the same length, equidistant from each other. $\varepsilon$ denotes the center-to-center distance between two segments. $\varepsilon_L = \frac{\varepsilon}{L}$ is supposed to be a small parameter without dimensions. In sections 2, 3, 4 and 5, we suppose that $L = 1$, so $\varepsilon = \varepsilon_L$. Let $l$, $\gamma_E$ and $\gamma_W$ be three positive real numbers. The fictitious boundaries are defined by:

$$\Gamma_E = \{ x \in \mathbb{R}^2 \text{ with } x_1 = \gamma_E \text{ and } 0 > x_2 > -l \}$$

and

$$\Gamma_W = \{ x \in \mathbb{R}^2 \text{ with } x_1 = \gamma_W \text{ and } 0 > x_2 > -l \}.$$ 

![Fig. 1. A simple pattern of an exhaust muffler.](image)

Let $\Omega$ be the domain composed by the muffler without the perforated wall, as shown on figure 2, such that $\Omega_{\varepsilon} = \Omega \setminus \Sigma_{\varepsilon}$. Finally let:

$$\Sigma = \{ x_2 = 0 \} \cap \Omega,$$

$$\Omega^+ = \{ x \in \Omega \text{ with } x_2 > 0 \},$$

and

$$\Omega^- = \{ x \in \Omega \text{ with } x_2 < 0 \}.$$
An important parameter for the design of mufflers is the porosity $\sigma \in [0, 1]$. By definition of the porosity, the length of the small segments is $(1 - \sigma)\varepsilon$, and the length of the holes is $\sigma\varepsilon$. A main assumption is that $\Sigma$ does not depend on $\varepsilon$. As a consequence:

$\sigma = \lim_{\varepsilon \to 0} \frac{|\Sigma| - |\Sigma_\varepsilon|}{|\Sigma|}$.

The total acoustic pressure $u_\varepsilon$ satisfies the Helmholtz equation in the domain $\Omega_\varepsilon$ and a homogeneous Neumann boundary condition on $\partial \Omega_\varepsilon \setminus (\Gamma_E \cup \Gamma_W)$. It satisfies moreover a transparent boundary condition on $(\Gamma_E \cup \Gamma_W)$, derived from the modal decomposition of the pressure in the tube (cf. Cutzach et al\textsuperscript{9}), that takes into account the effects of the outer part of the tube. So we have:

$$\begin{cases}
\Delta u_\varepsilon + k^2 u_\varepsilon = 0 & \text{in } \Omega_\varepsilon \\
\frac{\partial u_\varepsilon}{\partial n} = 0 & \text{on } \partial \Omega_\varepsilon \setminus (\Gamma_E \cup \Gamma_W) \\
\frac{\partial u_\varepsilon}{\partial n} = T_W(u_\varepsilon) + f_W & \text{on } \Gamma_W \\
\frac{\partial u_\varepsilon}{\partial n} = T_E(u_\varepsilon) + f_E & \text{on } \Gamma_E
\end{cases}$$  \hspace{1cm} (2.1)

where $f_E$ and $f_W$ are given by the incident wave. $T_E$ and $T_W$ are the Steklov-Poincaré operators mapping $H^{1/2}(\Gamma_E)$ on $H^{-1/2}(\Gamma_E)$ (resp. $H^{1/2}(\Gamma_W)$ on $H^{-1/2}(\Gamma_W)$). If the wave number $k$ is supposed to be such that $k < \frac{\pi}{L}$, there exists in the tube only one propagating mode. This assumption corresponds to the range of frequencies where the study of mufflers is relevant.

More explicitly, the expression of these operators is then given by:

$$T_E(u_{[\Gamma_E]}) = ik u_0(\gamma_E) \varphi_0 - \sum_{m \geq 1} \sqrt{\lambda_m - k^2} u_m(\gamma_E) \varphi_m$$  \hspace{1cm} (2.2)

$$T_W(u_{[\Gamma_W]}) = ik u_0(\gamma_W) \varphi_0 - \sum_{m \geq 1} \sqrt{\lambda_m - k^2} u_m(\gamma_W) \varphi_m$$  \hspace{1cm} (2.3)

where for $m \in \mathbb{N}$:
\( u_m(\gamma_E) = (u(\gamma_E, \cdot), \varphi_m)_{L^2(\Gamma_E)} \)

\( u_m(\gamma_W) = (u(\gamma_W, \cdot), \varphi_m)_{L^2(\Gamma_W)} \)

with:

\[ \varphi_0(x_2) = \frac{1}{\sqrt{I}} \]

and for \( m > 0 \):

\[ \lambda_m = \left( \frac{m \pi}{l} \right)^2 \]

\[ \varphi_m(x_2) = \sqrt{\frac{2}{l}} \cos(\sqrt{\lambda_m}x_2) \]

Using Fredholm theory, it may be proved that problem (2.1) is well posed, except for some resonance values of \( k \) (cf. Bonnet-Bendhia et al\(^{10}\)).

3. The asymptotic expansions

We are going to use a method which combines the multi-scales method of homogenization theory, and the matched asymptotic expansions method. The matched asymptotic expansions technique is needed because of the presence of a boundary layer in the neighborhood of \( x_2 = 0 \). Using formal calculations, we derive the problems solved by each term of the inner and the outer expansion (cf. Sanchez et al\(^{1}\), Bergman et al\(^{2}\), Sanchez\(^{11}\)).

3.1. The outer expansion

We assume that the solution of problem (2.1) can be expressed away from the boundary layer as an expansion in the domain \( \Omega^+ \cup \Omega^- \):

\[ u_\varepsilon = u_0(x) + \varepsilon u_1(x) + ... \quad (3.7) \]

More precisely, we have:

\[
\begin{cases}
 u_\varepsilon = u_0^+(x) + \varepsilon u_1^+(x) + ... \text{ in } \Omega^+ \\
 u_\varepsilon = u_0^-(x) + \varepsilon u_1^-(x) + ... \text{ in } \Omega^-
\end{cases} \quad (3.8)
\]

where \( u_j^+ \) (the restriction of \( u_j \) to \( \Omega^+ \)) and \( u_j^- \) (the restriction of \( u_j \) to \( \Omega^- \)) are two independent functions. The relationship between these two functions, will be established later on, after matching each of them with the boundary layer expansion. Such expansion, where the \( u_j \) are chosen as functions of \( x \) and not of \( \frac{x}{\varepsilon} \) means that, far from the boundary layer, fast variations of \( u_\varepsilon \) are negligible.
Substituting the solution by its expansion (3.7) in problem (2.1), we can show that the term $u_0$ solves the following problem:

$$
\begin{align*}
\Delta u_0 + k^2 u_0 &= 0 \quad \text{in } \Omega^+ \cup \Omega^- \\
\frac{\partial u_0}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus (\Gamma_E \cup \Gamma_W) \\
\frac{\partial u_0}{\partial n} &= T_E(u_0) + f_E \quad \text{on } \Gamma_E \\
\frac{\partial u_0}{\partial n} &= T_W(u_0) + f_W \quad \text{on } \Gamma_W
\end{align*}
$$

(3.9)

and furthermore that for each $j \geq 1$, the term $u_j$ solves:

$$
\begin{align*}
\Delta u_j + k^2 u_j &= 0 \quad \text{in } \Omega^+ \cup \Omega^- \\
\frac{\partial u_j}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus (\Gamma_E \cup \Gamma_W) \\
\frac{\partial u_j}{\partial n} &= T_E(u_j) \quad \text{on } \Gamma_E \\
\frac{\partial u_j}{\partial n} &= T_W(u_j) \quad \text{on } \Gamma_W
\end{align*}
$$

(3.10)

### 3.2. The inner expansion

In the following, we neglect the effects caused by the extremities of the perforated duct. We introduce a new system of coordinates $(O, y_1, y_2)$, in which the image of $\Sigma_\varepsilon$ by the mapping $x \mapsto y = \frac{x}{\varepsilon}$ becomes, as $\varepsilon$ tends to zero, a boundary $\Sigma_1$ which is infinite and periodic with respect to $y_1$, with period 1. $\Sigma_1$ is composed by small segments of length $(1 - \sigma)$. We consider in the following the domain $G = \{(0, 1|\times \mathbb{R}) \setminus \Gamma$ with $\Gamma = \{y \in \mathbb{R}^2$ with $\frac{\sigma}{2} < y_1 < 1 - \frac{\sigma}{2}$ and $y_2 = 0\}$. $G$ is a period of the domain study in the new frame.

The function $u_\varepsilon$ is assumed to expand in the vicinity of the perforated wall $\Sigma_\varepsilon$ as follows:

$$
 u_\varepsilon = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots + \varepsilon^m v_m + \ldots 
$$

(3.11)

where each term $v_j$ does not depend on $\varepsilon$, and satisfies:
Fig. 3. The cell G.

\[
\begin{aligned}
\begin{cases}
 v_j = v_j(x_1, y) ; y = (y_1, y_2) = \frac{x}{\varepsilon} \text{ and } x = (x_1, x_2) \\
v_j(x_1, y) \text{ is } 1 - \text{periodic w.r.t. } y_1 \\
v_j(x_1, .) \in S'(\infty) \cup S'(-\infty)
\end{cases}
\end{aligned}
\]

(3.12)

The sets \( S'(\infty) \) and \( S'(-\infty) \) are defined in appendix A. The local variable \( y_1 \) is related to the periodicity effects, while \( y_2 \) takes into account the boundary layer effects.

If we set \( v^\varepsilon(x_1, x_2) = v(x_1, x_2, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \), we have the following relations:

\[
\frac{\partial v^\varepsilon}{\partial x_i} = \frac{\partial v}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial v}{\partial y_i}
\]

(3.13)

\[
\Delta v^\varepsilon = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \frac{1}{\varepsilon} \Delta_y \frac{v}{\varepsilon} + \frac{2}{\varepsilon} \frac{\partial^2 v}{\partial x_1 \partial y_1} + \frac{2}{\varepsilon} \frac{\partial^2 v}{\partial x_2 \partial y_2}
\]

(3.14)

with

\[
\Delta_y = \sum_{i=1}^{2} \frac{\partial^2}{\partial y_i^2}
\]

Applying these operators to \( v_j \) (which do not depend on \( x_2 \)), we get:
\[
\Delta v_j = \frac{\partial^2 v_j}{\partial x_1^2} + \frac{1}{\varepsilon^2} \Delta_y v_j + \frac{2}{\varepsilon} \frac{\partial^2 v_j}{\partial x_1 \partial y_1}
\]  
(3.15)

Writing that the inner expansion satisfies the Helmholtz equation, we obtain at orders \(-2\), \(-1\) and 0, the following equations, where \(x_1\) plays the role of a parameter:

\[
\Delta_y v_0 = 0, \text{ in } G
\]  
(3.16)

\[
\Delta_y v_1 + 2 \frac{\partial^2 v_0}{\partial x_1 \partial y_1} = 0, \text{ in } G
\]  
(3.17)

\[
\Delta_y v_2 + 2 \frac{\partial^2 v_1}{\partial x_1 \partial y_1} + \frac{\partial^2 v_0}{\partial x_1^2} + k^2 v_0 = 0, \text{ in } G
\]  
(3.18)

From the boundary condition on \(\Sigma_\varepsilon\) and taking into account that \(v_j\) does not depend on \(x_2\), we derive:

\[
\frac{\partial v_0}{\partial y_2} = 0, \quad \frac{\partial v_1}{\partial y_2} = 0 \quad \text{and} \quad \frac{\partial v_2}{\partial y_2} = 0 \text{ on } \Gamma.
\]

In order to define more precisely the asymptotic behavior of the terms of the inner expansion, and to complete the problems of each term of the outer expansion we now need to match both the expansions. We shall present in the following subsection some matching rules.

### 3.3. Matching the expansions

To ensure the coherence and the unicity of inner and outer expansions we have to establish some matching rules. The first matching rule between the inner and the outer expansion will concerns the zero order, and we can exprime it as follows:

\[
\text{outer limit of inner limit = inner limit of outer limit}
\]  
(3.19)

In the following, \(x_1\) and \(y_1\) are considered as parameters, and limits concerns only the variables \(x_2\) and \(y_2\). The inner (resp. outer) limit means the limit for a fixed value of \(y_2\) (resp. \(x_2\)) when \(\varepsilon\) tends towards zero, of the inner (resp. outer) expansion.

Therefore, \(v_0(x_1, y_1, y_2)\) is the inner limit of \(u_\varepsilon\). In the same way, \(u_0(x_1, x_2)\) is the outer limit of \(u_\varepsilon\). By substituting \(y_2\) by \(\frac{x_2}{\varepsilon}\) in \(v_0(x_1, y_1, y_2)\), and \(x_2\) by \(\varepsilon y_2\) in \(u_0(x_1, x_2)\), the previous rule leads finally, when \(\varepsilon\) tends towards zero, to:

\[
\begin{cases}
  u_0(x_1, 0^+) = v_0(x_1, y_1, +\infty) \\
  \text{and} \\
  u_0(x_1, 0^-) = v_0(x_1, y_1, -\infty)
\end{cases}
\]  
(3.20)
To define a transmission condition for the first term of the outer expansion, a matching rule for higher levels can be derived according to the following formula:

\[
\text{outer expansion 2 terms of inner expansion 2 terms} = \text{inner expansion 2 terms of outer expansion 2 terms}
\]

or according the next one, if we have to define the transmission condition for normal derivatives of the same term:

\[
\text{outer expansion 2 terms of inner expansion 3 terms} = \text{inner expansion 3 terms of outer expansion 2 terms}
\]

The inner expansion (resp. outer) means the inner (resp. outer) expansion of \( u_\varepsilon \) for a fixed value of \( y_2 \) (resp. \( x_2 \)). The number of terms shows how much terms we keep in the expansion. These rules will be detailed later.

4. The problem solved by \( u_0 \)

We have determined in the previous sections the partial differential equations solved by \( u_0 \) in each subdomain \( \Omega^+ \) and \( \Omega^- \), and the boundary conditions on the external boundaries of \( \Omega \). It remains to define transmission conditions on the interface \( \Sigma \), between \( \Omega^+ \) and \( \Omega^- \).

4.1. Transmission condition for \( u_0 \)

We begin with the study of the asymptotic behavior of \( v_0 \). We show first, that \( v_0 \) depends only on \( x_1 \). Indeed, the zero order term \( v_0 \) of the asymptotic inner expansion solves:

\[
\begin{align*}
\Delta v_0 &= 0 \quad \text{in } G \\
\frac{\partial v_0}{\partial y_2} &= 0 \quad \text{on } \Gamma
\end{align*}
\]

Applying Lemma A.1 of appendix A, we obtain the asymptotic behavior of any solution of problem (4.23):

\[
v_0(x_1, y) = \alpha_0^+ y_2 + \beta_0^+ + v_{res}^+(x_1, y) \quad \text{for large } |y_2|
\]

where \( v_{res}^+(x_1, .) \in S(+\infty) \) and \( v_{res}^-(x_1, .) \in S(-\infty) \). These sets are defined in appendix A. We match now the expansions, according to the rule (3.20). If \( \alpha_0^0 \neq 0 \), the expansion for a fixed \( x_2 \) when \( \varepsilon \) tends towards zero, of the inner limit \( v_0(x_1, x_1/\varepsilon, y_2 = x_2/\varepsilon) \) gives in the right hand side of (3.20), the expression:

\[
\lim_{\varepsilon \to 0} (\alpha_0 x_2/\varepsilon)
\]

where \( \alpha = \alpha_0^+ \) for \( x_2 > 0 \) and \( \alpha = \alpha_0^- \) if \( x_2 < 0 \).
On the other hand, the expansion for a fixed $y_2$, when $\varepsilon$ tends towards zero, of the outer limit $u_0(x_1, x_2 = \varepsilon y_2)$ gives in the left hand side of (3.20) the expression:

$$
\begin{cases}
  u_0(x_1, 0^+) = u_0^+(x_1, 0) & \text{if } y_2 > 0 \\
  u_0(x_1, 0^-) = u_0^-(x_1, 0) & \text{if } y_2 < 0
\end{cases}
$$

(4.26)

One only has to identify (4.25) with (4.26). As a consequence we have $\alpha_0^+ = \alpha_0^- = 0$, otherwise $u_0$ would tend towards infinity at $x_2 = 0$. Going back to (4.24), this proves that $\frac{\partial v_0}{\partial y_i} \in L^2(G)$.

Finally, it results from theorem A.1 of appendix A that problem (4.23) has a unique solution, which is the vanishing solution, up to a constant with respect to $y$.

Summing up, the fitting with the boundary layer gives the continuity of $u_0$ over $\Sigma$:

$$
u_0^+(x_1, 0) = u_0^-(x_1, 0) = v_0(x_1)$$

(4.27)

Consequently we have determined $v_0$, that is constant with respect to $y$, and depends only on the parameter $x_1$. To complete the problem satisfied by $u_0$, it remains to find a condition satisfied by its normal derivative on $\Sigma$.

4.2. Transmission condition for the normal derivative of $u_0$

First we have to define the asymptotic behavior of the first order term in the inner expansion. It solves the following equations:

$$
\begin{cases}
  \Delta_y v_1 = 0 & \text{in } G \\
  \frac{\partial v_1}{\partial y_2} = 0 & \text{on } \Gamma
\end{cases}
$$

(4.28)

By the same argument, that gives the asymptotic behavior of $v_0$, we obtain an analogous result for $v_1$ as follows:

$$v_1(x_1, y) = \alpha_1^\pm y_2 + \beta_1^\pm + v_{res}^\pm(x_1, y) \text{ for large } |y_2|$$

(4.29)

where $v_{res}^+(x_1, \cdot) \in S(\pm\infty)$ and $v_{res}^-(x_1, \cdot) \in S(-\infty)$.

If we apply the Green formula in $B = G \cap ([0, 1[ - \xi, \xi])$, for a sufficiently large $\xi$ and taking into account the periodicity of $v_1$ with respect to $y_1$, we can deduce:

$$
\int_B \Delta v_1 dy = \int_{\partial B} \frac{\partial v_1}{\partial n} d\gamma = 0
$$

Hence, according to (4.29):

$$
\int_{y_2 = +\xi} \left( \alpha_1^+(x_1) + \frac{\partial v_{res}^+}{\partial n}(x_1, y) \right) dy_1 + \int_{y_2 = -\xi} \left( -\alpha_1^-(x_1) + \frac{\partial v_{res}^-}{\partial n}(x_1, y) \right) dy_1 = 0
$$
If \( \xi \) tends to infinity, we obtain:

\[
\alpha_1^+(x_1) = \alpha_1^-(x_1)
\]

Let us apply now the second matching rule to \( u_0 \), according to (3.21). Keeping only the zero and first order terms with respect to \( \varepsilon \), we obtain the identity:

\[
v_0(x_1) + \varepsilon v_1(x_1, y_1, \frac{x_2}{\varepsilon}) + O(\varepsilon^2) = u_0(x_1, x_2) + \varepsilon u_1(x_1, x_2) + O(\varepsilon^2)
\]

(4.30)

We replace in the left hand side \( v_1 \) by its asymptotic behavior at infinity, in the right hand side we replace \( u_0 \) by its Taylor expansion of the first order at \( x_2 = 0 \):

\[
\begin{align*}
&\begin{cases}
  v_0(x_1) + \varepsilon (\alpha_1(x_1)y_2 + \beta_1(x_1)) + O(\varepsilon^2) = \\
  u_0(x_1, 0) + x_2 \frac{\partial u_0}{\partial x_2}(x_1, 0) + \varepsilon u_1(x_1, x_2) + O(\varepsilon^2)
\end{cases} \\
\end{align*}
\]

(4.31)

If we substitute \( y_2 \) by \( \frac{x_2}{\varepsilon} \) and bring together the terms of same order in \( \varepsilon \), it follows that:

\[
\begin{align*}
&\begin{cases}
  (v_0(x_1) + \alpha_1(x_1)x_2 + \varepsilon \beta_1(x_1)) + O(\varepsilon^2) = \\
  (u_0(x_1, 0) + x_2 \frac{\partial u_0}{\partial x_2}(x_1, 0) + \varepsilon u_1(x_1, x_2) + O(\varepsilon^2)
\end{cases} \\
\end{align*}
\]

(4.32)

Therefore by identification of terms of different orders and if we take into account equation (4.23), we obtain:

\[
\begin{align*}
&\begin{cases}
  \frac{\partial u_0^+}{\partial x_2}(x_1, 0) = \alpha_1^+(x_1), \quad \frac{\partial u_0^-}{\partial x_2}(x_1, 0) = \alpha_1^-(x_1) \\
  u_1^+(x_1, 0) = \beta_1^+(x_1), \quad u_1^-(x_1, 0) = \beta_1^-(x_1)
\end{cases} \\
\end{align*}
\]

(4.33)

But \( \alpha_1^+ = \alpha_1^- \), from which one deduces the continuity of the normal derivative of \( u_0 \) across \( \Sigma \).

According to (4.27) and the previous result, it follows finally that:

\[
[u_0]_{\Sigma} = 0 \quad \text{and} \quad \left[\frac{\partial u_0}{\partial x_2}\right]_{\Sigma} = 0.
\]

Hence, we have defined the problem solved by the zero order term of the outer expansion. Indeed, \( u_0 \in H^1(\Omega) \) solves the following problem:
\[
\begin{align*}
\Delta u_0 + k^2 u_0 &= 0 \quad \text{in } \Omega \\
\frac{\partial u_0}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus (\Gamma_E \cup \Gamma_W) \\
\frac{\partial u_0}{\partial n} &= T_E(u_0) + f_E \quad \text{on } \Gamma_E \\
\frac{\partial u_0}{\partial n} &= T_W(u_0) + f_W \quad \text{on } \Gamma_W
\end{align*}
\] (4.34)

**Remark 4.1.** The zero order approximation \(u_0\) solves a problem similar to (2.1) where the perforated duct \(\Sigma_e\) has been removed. This approximation "do not see" the perforated duct.

5. The problem solved by \(u_1\)

By the same way, we propose to complete the problem (3.10) solved by \(u_1\) with transmission conditions on the boundary \(\Sigma\).

5.1. Transmission condition for \(u_1\)

We established in (4.33) that:

\[u_1^{+}(x_1, 0) = \beta_1^{+}(x_1) \text{ and } u_1^{-}(x_1, 0) = \beta_1^{-}(x_1),\]

where \(\beta_1^{+}\) and \(\beta_1^{-}\) are defined by the asymptotic behavior of the function \(v_1\) in (4.29). Let consider the following problem:

\[
\begin{align*}
\Delta \phi &= 0 \quad \text{in } G \\
\frac{\partial \phi}{\partial y_2} &= 0 \quad \text{on } \Gamma \\
\phi \text{ is 1-periodic w.r.t. } y_1 \\
\phi \text{ is equivalent to } (y_2 + \beta^+) \text{ for some } \beta^+, \text{ when } y_2 \to +\infty \\
\phi \text{ is equivalent to } (y_2 + \beta^-) \text{ for some } \beta^-, \text{ when } y_2 \to -\infty \\
\frac{\partial \phi}{\partial y_1} \in L^2(G)
\end{align*}
\] (5.35)
We have then the following theorem:

**Theorem 5.1.** Problem (5.35) has a unique solution up to an additive constant.

**Proof.** Let \( b \) the function defined by:

\[
b(y_2) = \theta(y_2) y_2 \text{ on } \mathbb{R}
\]

where \( \theta \in C^\infty(\mathbb{R}) \) is such that for a sufficiently large \( \xi \), \( \theta = 0 \) in \( \frac{-\xi}{2}, \frac{\xi}{2} \), and \( \theta = 1 \) for \( |y_2| > |\xi| \). For any solution \( \phi \) of (5.35), the function \( w = \phi - b \) belongs to the functional space \( V \) defined in appendix A and solves the following problem:

\[
\begin{align*}
\Delta_y w &= F \quad \text{in } G \\
\frac{\partial w}{\partial n} &= 0 \quad \text{on } \Gamma
\end{align*}
\]

where \( F = -\Delta_y b \). Moreover, \( F \) satisfies the hypothesis of theorem A.1 and the compatibility condition \( \int_G Fdy = 0 \).

According to theorem A.1, problem (5.37) has a unique solution \( w \in V \). And so \( \phi \) is unique up to an additive constant. Notice that the value of \( \beta^+ - \beta^- \) is therefore uniquely defined. \( \square \)

We can now derive the expression of \( v_1 \). Since \( \alpha_1^+(x_1) = \alpha_1^-(x_1) = \frac{\partial u_0}{\partial x_2}(x_1,0) \), it follows by linearity that:

\[
v_1(x_1, y) = \frac{\partial u_0}{\partial x_2}(x_1,0) \phi(y) + C(x_1)
\]

where \( C \) is a complex constant depending only on \( x_1 \).

As a consequence,

\[
[u_1] = \beta_1^+(x_1) - \beta_1^-(x_1) = \frac{\partial u_0}{\partial x_2}(x_1,0)(\beta^+ - \beta^-)
\]

The value of \( \beta^+ - \beta^- \) is explicitly determined in Morse et al\(^{12} \) by using techniques of complex variable:

\[
\beta^+ - \beta^- = \frac{2}{\pi} \ln \left( \frac{\delta}{2} + \frac{1}{2\delta} \right)
\]

with \( \delta = tg\left( \frac{\pi}{4} \sigma \right) \) (remember that \( \sigma \) denotes the porosity of the perforated duct).
5.2. Transmission condition for the normal derivative of $u_1$

To compute the function $u_1$, it remains to evaluate $\left[\frac{\partial u_1}{\partial x_2}\right]_{x_2}$. We then take into account the asymptotic behavior of $v_2$, the second order term of the inner expansion. It solves the following problem:

$$
\begin{align*}
-\Delta_y v_2 &= 2\frac{\partial^2}{\partial x_1 \partial y_1} v_1 + \frac{\partial^2}{\partial x_1^2} v_0 + k^2 v_0 & \text{in } G \\
\frac{\partial v_2}{\partial y_2} &= 0 & \text{on } \Gamma \\
v_2 &\text{ is 1-periodic w.r.t. } y_1 \\
\frac{\partial v_2}{\partial y_1}(x_1,\cdot) &\in L^2(G)
\end{align*}
$$

Let us remind that $\frac{\partial^2}{\partial x_1 \partial y_1} v_1(x_1,\cdot) \in S(+\infty) \cup S(-\infty)$. We set:

$$
\psi = v_2 + 1/2(\frac{\partial^2}{\partial x_1} v_0 + k^2 v_0) y_2^2
$$

So:
We can apply lemma A.1 in appendix A. It follows that for large $|y_2|$, we have:

$$v_2(x_1, y) = -1/2 \left( \frac{\partial^2 v_0}{\partial x_1^2} + k^2 v_0 y_2^2 + \alpha_2^+ (x_1) y_2 + \beta_2^+ (x_1) + v_{res}^+(x_1, y) \right),$$

where $v_{res}^+(x_1, \cdot) \in S(+\infty)$ and $v_{res}^-(x_1, \cdot) \in S(-\infty)$.

Let us apply now the third matching rule (3.22), which allows to determine the transmission condition satisfied by the normal derivative of $u_1$. We keep for both the inner and outer expansions, all terms of zero, first and second order with respect to $\varepsilon$. We obtain for the inner expansion:

$$u_\varepsilon = v_0(x_1) + \varepsilon v_1(x_1, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) + \varepsilon^2 v_2(x_1, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) + \ldots$$

We substitute $v_0$ and $v_1$ by their asymptotic behavior w.r.t. $y_2 = \frac{x_2}{\varepsilon}$:

$$u_\varepsilon(x) = v_0(x_1) + \varepsilon(\alpha_1 x_2 / \varepsilon + \beta_1 + \ldots) + \varepsilon^2 \left( -\frac{\partial^2 v_0}{\partial x_1^2} + k^2 v_0 \frac{x_2^2}{2\varepsilon^2} + \alpha_2 x_2 / \varepsilon + \beta_2 + \ldots \right) + \ldots$$

In the right hand side of the equation obtained with the matching rule (3.22), we substitute $u_0$ by its second order Taylor expansion in $x_2 = 0$, and $u_1$ by its first order Taylor expansion. Finally, it remains to identify both following expressions:

$$v_0(x_1) + \alpha_1 x_2 - \frac{1}{2} \left( \frac{\partial^2 v_0}{\partial x_1^2} + k^2 v_0 \right) \frac{x_2^2}{2} + \varepsilon(\alpha_2 x_2 + \beta_1) + O(\varepsilon^2)$$

(5.42)
and

\[
\begin{aligned}
u_0(x_1,0) + x_2 \frac{\partial u_0}{\partial x_2}(x_1,0) + \frac{1}{2} x_2 \frac{\partial^2 u_0}{\partial x_2^2}(x_1,0) \\
+ \varepsilon \left( u_1(x_1,0) + x_2 \frac{\partial u_1}{\partial x_2}(x_1,0) \right) + O(\varepsilon^2).
\end{aligned}
\]

If we take into account that \( u_0(x_1,0) = v_0(x_1) \) and \( \frac{\partial u_0}{\partial x_2}(x_1,0) = \alpha_1 \), the identification of the zero order terms leads to:

\[
\frac{\partial^2}{\partial x_1^2}v_0 + k^2 v_0 + \frac{\partial^2 u_0}{\partial x_2^2}(x_1,0) = 0
\]

It follows that:

\[
\frac{\partial u_1^+}{\partial x_2}(x_1,0) = \alpha_2^+(x_1); \quad \frac{\partial u_1^-}{\partial x_2}(x_1,0) = \alpha_2^-(x_1)
\]

Consequently:

\[
\frac{\partial u_1^+}{\partial x_2}(x_1,0) - \frac{\partial u_1^-}{\partial x_2}(x_1,0) = \alpha_2^+(x_1) - \alpha_2^-(x_1)
\]

(5.44)

To conclude, it remains to determine the value of the function \( \alpha_2^+ - \alpha_2^- \). It’s the purpose of the following lemma:

**Lemma 5.1.** Let \( \psi \) be a solution of (5.41). Then it has an asymptotic behavior of the form

\[
\alpha_2^+(x_1)y_2 + \beta_2^+(x_1) + v_{res}^+(x_1,y) \quad \text{when} \quad y_2 \to \pm \infty
\]

where \( v_{res}^+(x_1,\cdot) \in S(+\infty) \) and \( v_{res}^-(x_1,\cdot) \in S(-\infty) \) and

\[
\alpha_2^+(x_1) - \alpha_2^-(x_1) = 0.
\]

**Proof.** Applying the Green formula in the domain \( B = G \cap ([0,1[\times \xi,\zeta]) \) and taking into account the properties of the right hand side of problem (5.41), we get:

\[
\int_B \frac{\partial^2}{\partial x_1 \partial y_1}v_1(x_1,y)dy = 0.
\]

It follows that:

\[
\alpha_2^+(x_1) - \alpha_2^-(x_1) = (\frac{\partial^2}{\partial x_1^2}v_0(x_1) + k^2v_0(x_1)) \int_{\Gamma} y_2n_2ds = 0
\]

from which we conclude the final result of the lemma \( \square \).
The proof of the existence of $v_2$ is a consequence of the following theorem, which ensures the existence of $\psi$.

**Theorem 5.2.** There exists $\psi$ in $V$ that solves (5.41).

**Proof.** Consider $b$, the function defined by (5.36) and set $\alpha_2 = \alpha_2^+ = \alpha_2^-$. Similarly to theorem 5.1, we set $w = \psi - \alpha_2 b$. The function $w$ solves problem (5.37) defined in theorem 5.1, where $F = \alpha_2 \Delta y b + \frac{\partial^2}{\partial x_1 \partial y_1} v_1$. The function $F$ satisfies the hypothesis of theorem A.1 in appendix A, which ensures that there exists only one solution $w \in V$ up to an additive constant $\Omega$.

Finally, the first order term $u_1$ solves the following problem:

$$
\begin{align*}
\Delta u_1 + k^2 u_1 &= 0 \quad \text{in } \Omega^+ \cup \Omega^-
\\
\frac{\partial u_1}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus (\Gamma_E \cup \Gamma_W)
\\
[u_1] &= (\beta^+ - \beta^-) \frac{\partial u_0}{\partial x_2} \quad \text{on } \Sigma
\\
\left[ \frac{\partial u_1}{\partial n} \right] &= 0 \quad \text{on } \Sigma
\\
\frac{\partial u_1}{\partial n} &= T_E(u_1) \quad \text{on } \Gamma_E
\\
\frac{\partial u_1}{\partial n} &= T_W(u_1) \quad \text{on } \Gamma_W
\end{align*}
$$

(5.45)

**6. Numerical study**

The numerical results are obtained by a finite element discretization with the MELINA code (cf. Martin¹³). In the next section we will show two different ways to evaluate a first order approximation of $u_\varepsilon$.

**6.1. The problem solved by the first order approximation**

The numerical evaluation of the first order approximation of $u_\varepsilon$ leads to compute the two first terms of the outer expansion. Let $u_{\varepsilon,1}$ be the first order approximation of $u_\varepsilon$:

$$
u_{\varepsilon,1} = u_0 + \varepsilon u_1
$$

(6.46)
The computation of \( u_{\varepsilon,1} \) can be performed by two different ways. The first one consists in computing separately \( u_0 \), solution of (4.34) and \( w \) solution of the following problem:

\[
\begin{aligned}
\Delta w + k^2 w &= 0 \quad \text{in } \Omega^+ \cup \Omega^- \\
\frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial \Omega \setminus (\Gamma_E \cup \Gamma_W) \\
[w] &= \frac{\partial u_0}{\partial x_2} \quad \text{on } \Sigma \\
\left[ \frac{\partial w}{\partial n} \right] &= 0 \quad \text{on } \Sigma \\
\frac{\partial w}{\partial n} &= T_E(w) \quad \text{on } \Gamma_E \\
\frac{\partial w}{\partial n} &= T_W(w) \quad \text{on } \Gamma_W
\end{aligned}
\]  

(6.47)

The first order approximation \( u_{\varepsilon,1} \) is then obtained by the following expression:

\[
u_{\varepsilon,1} = u_0 + \varepsilon (\beta^+ - \beta^-) w.
\]  

(6.48)

The advantages of this first method are:

1) The finite element matrix is the same for both problems (4.34) and (6.47).

2) These problems do not depend from the values of \( \sigma \) and \( \varepsilon \). Hence, if we change one of these parameters, we have only to evaluate the expression (6.48). The disadvantages are that:

1) The right hand side of problem (6.47) has a strong singularity at the extremities of \( \Sigma \). In fact the behavior of \( u_0 \) is affected by the presence of non convex corners where \( \frac{\partial u_0}{\partial x_2} \sim r^{-1/3} \) (cf. Grisvard\textsuperscript{12}), where \( r \) denotes the distance to the corner. It follows that \( \int_{\Omega_c} |\nabla u_{\varepsilon,1}|^2 = +\infty \), which means that the local energy of this approximate solution is infinite. This effect is not physically realistic but is an artificial consequence of the homogenization method.

2) We have to solve two problems to evaluate \( u_{\varepsilon,1} \).

The second way to evaluate a first order approximation of \( u_\varepsilon \) consists in determining a problem satisfied by \( u_{\varepsilon,1} \) at order 1. The function \( u_0 \) solves problem (4.34) and \( u_1 \) solves problem (5.45). Hence, \( u_{\varepsilon,1} \) satisfies the following equation on \( \Sigma \):

\[
[u_{\varepsilon,1}] = \varepsilon (\beta^+ - \beta^-) \frac{\partial u_0}{\partial x_2} = \varepsilon (\beta^+ - \beta^-) \frac{\partial u_{\varepsilon,1}}{\partial x_2} - \varepsilon^2 (\beta^+ - \beta^-) \frac{\partial u_1}{\partial x_2} \text{ on } \Sigma.
\]  

(6.49)

If we neglect in the previous equation the second order term and if we call \( \tilde{u}_{\varepsilon,1} \) the solution of the modified equations, we obtain the following transmission problem:
muffler’s transmission losses

\[
\begin{align*}
\Delta \tilde{u}_{\varepsilon,1} + k^2 \tilde{u}_{\varepsilon,1} &= 0 & \text{in } \Omega^+ \cup \Omega^- \\
\frac{\partial \tilde{u}_{\varepsilon,1}}{\partial n} &= 0 & \text{on } \partial \Omega \setminus (\Gamma_E \cup \Gamma_W) \\
\left[ \frac{\partial \tilde{u}_{\varepsilon,1}}{\partial n} \right] &= 0 & \text{on } \Sigma \\
\tilde{u}_{\varepsilon,1} &= \varepsilon(\beta^+ - \beta^-) \frac{\partial \tilde{u}_{\varepsilon,1}}{\partial x_2}(x_1,0) & \text{on } \Sigma \\
\frac{\partial \tilde{u}_{\varepsilon,1}}{\partial n} &= T_E(\tilde{u}_{\varepsilon,1}) + f_E & \text{on } \Gamma_E \\
\frac{\partial \tilde{u}_{\varepsilon,1}}{\partial n} &= T_W(\tilde{u}_{\varepsilon,1}) + f_W & \text{on } \Gamma_W
\end{align*}
\]  

(6.50)

In the previous problem we have an impedance condition on the boundary \( \Sigma \), which takes into account the perforated wall. Compared to the first method, the disadvantage of this formulation is that we have to solve the problem for each value of \( \varepsilon(\beta^+ - \beta^-) \). The advantages are that for a given value of \( \varepsilon(\beta^+ - \beta^-) \), we need to solve only one problem and we eliminate the weak regularity of the right hand side of the system. For these reasons, in the following section, all the numerical results corresponding to the first order approximation of \( u_\varepsilon \) are obtained by the second formula.

6.2. Numerical results

In the following we compare the values of the transmission losses computed for the first order approximation \( u_{\varepsilon,1} \) to the values obtained by the computation of \( u_\varepsilon \) and to those obtained by the zero order approximation \( u_0 \). The finite element solution of (2.1) will be referred as the “exact computation”. Let us remind that the computation of \( u_\varepsilon \) leads to use a finite element mesh refined around the perforated boundary \( \Sigma_\varepsilon \) (figure 5) while we only have to use a regular mesh to compute \( u_{\varepsilon,1} \) (figure 6). The transmission losses \( TL \) is defined by the following expression:

\[
TL = 20 \log \left| \frac{u^{\text{inc}}_{\varepsilon,1}}{|p|_{\varepsilon,1}} \right|
\]  

(6.51)

where \( u^{\text{inc}} \) is the incident plane wave and \( p \) refers to the pressure field \( (u_\varepsilon, u_{\varepsilon,1} \) or \( u_0 ) \).

We set \( k \) the wave number, \( L \), \( S_1 \) and \( S_2 \) are respectively the length, the small and the big section of the exhaust muffler. \( \varepsilon \) denotes the center-to-center distance between two segments of the perforated duct. In the following, the numerical results are obtained for \( S_2/L = 4/8.5, S_1/L = 1/8.5 \) for several values of \( \sigma, \varepsilon \) and \( KL = k \ast L \). We set \( \varepsilon = 8.5 \frac{\varepsilon}{L} \).
where $\varepsilon_L = \frac{\tilde{\varepsilon}}{L}$ is supposed to be the small parameter. $\hat{h}$ denotes the Finite Element mesh size, and $h = 8.5 \frac{\hat{h}}{L}$ a parameter without dimensions, used to characterize meshes.

6.2.1. Accuracy of the zero and first order approximation

In figures 7, 8 and 9, we compare the transmission losses $TL$ computed for the function $u_\varepsilon$ (exact computation), $u_{\varepsilon,1}$ (the first order approximation) and $u_0$ (the zero order approximation). The porosity $\sigma$ is equal to 0.5. The three figures correspond respectively to ($\varepsilon = 1$ and $h = 0.125$), ($\varepsilon = 0.5$ and $h = 0.0833$) and ($\varepsilon = 0.25$ and $h = 0.0625$). These figures confirm the accuracy of the first order approximation. In fact the values of frequencies where there is a high attenuation are not detected at the zero order approximation.

6.2.2. Effects of mesh refining in exact and in homogenized computation

In this paragraph, we shall point out the fact that the first order approximation, using homogenization, does not need such a fine mesh as the Finite Element computation on the exact geometry does.

For the first order computation, as well as the Finite Element method on the exact geometry (figure 10 and 11), a reference solution will be provided by the Finite Element computation using the finest mesh ($h = 0.0625$). The geometry corresponds to a muffler with a perforated duct for $\varepsilon = 1$.

Figure 10 is obtained by plotting the relative error on the transmission losses obtained by the first order approximation for several meshes. It turns out that the error decreases with respect to the mesh size, though staying below a $4.10^{-4}$ rate.

The Finite Element computation on the exact geometry is much more sensitive to the mesh size as shown in figure 11. Actually, a too coarse mesh leads to non significant results. A threshold is observed at $h = 0.25$, meaning 4 elements per period ($\varepsilon = 1$), raising the error on the transmission losses up to 0.6 rate, for high frequencies.

Figure 12 and 13 show that a homogenized computation with the mesh corresponding to $h = 0.25$ still provides accurate results when $\varepsilon$ decreases. In figure 12 (respectively figure 13), for $\varepsilon = 0.5$ (respectively $\varepsilon = 0.25$) the reference solution is provided by finite element mesh with $h = 0.0625$ (respectively $h = 0.0417$).

As a conclusion, using first order homogenized computation with reasonably fine mesh turns out to represent an effective alternative to a computation on the exact geometry. The mesh should take into account the wavelength in a precise way.

6.2.3. Variation of $TL$ with respect to the porosity

In figure 14, the transmission loss of an exhaust muffler with a perforated duct is plotted for different values of the porosity $\sigma$. These curves are compared to the transmission losses
of the zero order approximation ($\sigma = 1$ and $\varepsilon = 0$). We can see on these curves how much the presence of a perforated duct can improve the transmission losses around the resonance frequencies of the exhaust muffler.

6.2.4. Comparison between the exact computation and the first order approximation

In figures 15, we present the isovalues of the relative error between the acoustic pressure obtained by a computation on the exact geometry and that obtained by using the first order approximation. We notice that the errors are not localized around $\Sigma$.

We present in figure 16 the relative error curves between the values of the acoustic pressure obtained by an exact computation and those obtained by using a first order approximation for different values of $\varepsilon$. We plot the errors on the upper part of $\Sigma$. The errors increase near the extremities of the perforated duct, which is related to the fact that the boundary effects are neglected in the homogenization method.

6.2.5. Convergence to the zero order approximation

In figure 17, the transmission loss of the muffler is plotted with respect to $KL$, for a single value of the porosity ($\sigma = 0.5$), for three values of the parameter $\varepsilon$. We remark that the transmission losses curves obtained by exact computation converge to that obtained by using the zero order approximation ($\sigma = 1$ and $\varepsilon = 0$) for $KL < 7$. 
Fig. 5. The mesh of the muffler with a perforated duct.

Fig. 6. The mesh of the homogenized geometry.
Fig. 7. Transmission losses for $\varepsilon = 1$ and $h = 0.12$.

Fig. 8. Transmission losses for $\varepsilon = 0.5$ and $h = 0.08$. 
porosity=0.5

Fig. 9. Transmission losses for $\varepsilon = 0.25$ and $h = 0.06$.

Curves obtained by first order approximation

Fig. 10. Relative error curves for different meshes.
Fig. 11. Relative error curves for different meshes.

Fig. 12. Relative error curves for different meshes for $\varepsilon = 0.5$. 
Curves obtained by first order approximation

Fig. 13. Relative error curves for different meshes for $\varepsilon = 0.25$.

Fig. 14. Variation of $TL$ w.r.t. the porosity for $\varepsilon = 1$. 
Fig. 15. Isovalues of the relative errors between exact computation and first order approximation for $\varepsilon = 1$.

Fig. 16. Relative error on $\Sigma^+$. 
Fig. 17. Convergence to the zero order approximation.

Fig. 18. Zoom in the neighborhood of the peaks.
Appendix A

Asymptotic behavior of harmonic periodic functions:

The results of the present section are basically those of Nguetseng\(^8\). Notice that the introduction of the weighted Sobolev space \( V \) makes the proof of theorem A.1 easier.

Consider two functions \( \psi_1 \in \mathcal{D}(\mathbb{R}) \) and \( \psi_2 \in C^\infty(\mathbb{R}) \) such that \( \psi_2(y) = 0 \) for \( y < -1 \) and \( \psi_2(y) = 1 \) for \( y > 1 \). Let us define:

- \( S(+\infty) \) the set of functions \( u : \mathbb{R} \rightarrow \mathbb{R} \), 1-periodic w.r.t. \( y_1 \), such that the function \( \psi_1(y_1)\psi_2(y_2)u \in S(\mathbb{R}^2) \).

- \( S'(+\infty) \) the set of functions \( u : \mathbb{R} \rightarrow \mathbb{R} \), 1-periodic w.r.t. \( y_1 \), such that the function \( \psi_2(y_2)u \) belongs to \( S'(\mathbb{R}^2) \).

\( S(-\infty) \) and \( S'(-\infty) \) are defined in a similar way. Let \( F \) be a function such that:

\[
F \in S(+\infty) \tag{A.1}
\]

\[
F \in L^2(\mathbb{R}); \quad F \text{ is } 1 - \text{periodic w.r.t } y_1. \tag{A.2}
\]

The next Lemma deals with the asymptotic behavior of terms of the inner expansion.

**Lemma A.1.** Let \( a > 0 \) and \( u \) be a function defined in \( \mathbb{R} \times ]a, +\infty[ \) and satisfying:

\[
\begin{cases}
-\Delta u = F & \text{in } \mathbb{R} \times ]a, +\infty[ \\
u \text{ is } 1 - \text{periodic w.r.t } y_1 & \quad \text{A.3.1} \\
u \in S'(+\infty) & \quad \text{A.3.2}
\end{cases}
\]

then there exist, \( \alpha^+, \beta^+ \in \mathbb{R} \) and \( u^+_{\text{res}} \in S(+\infty) \) such that:

\[
u(y) = \alpha^+y_2 + \beta^+ + u^+_{\text{res}}(y) \quad \text{when } y_2 \rightarrow +\infty \quad \text{A.4}\]

**Proof.** We deal first with the case of a zero right hand side. The function \( u \) is then developed under a Fourier expansion:

\[
\begin{align*}
u(y_1, y_2) &= \sum_{n \in \mathbb{Z}} a_n(y_2) \exp(\text{i}n2\pi y_1) \\
a_n(y_2) &= \int_0^1 \nu(y_1, y_2) \exp(-\text{i}n2\pi y_1) dy_1
\end{align*}
\]
Thanks to the smoothness of $u$, we can write:

$$
\Delta u(y_1, y_2) = \sum_{n \in \mathbb{Z}} \left( \frac{d^2 a_n}{dy_2^2}(y_2) - n^2 a_n(y_2) \right) \exp(i n 2 \pi y_1)
$$

Since $\Delta u(y_1, y_2) = 0$, we get for $n \in \mathbb{N}$:

$$
\frac{d^2 a_n}{dy_2^2}(y_2) - n^2 a_n(y_2) = 0
$$

and since $u \in S'(\infty)$, it follows that:

$$
a_0(y_2) = \alpha^+ y_2 + \beta^+ \quad \text{et} \quad a_n(y_2) = \exp(-|n| y_2)
$$

which acheives the proof in the homogeneous case. The general case requires the use of the method of variation of constant, and the previous result. Taking into account the asymptotic behavior of $F$, we show that the asymptotic behavior of $u$ is similar to that in the homogenous case. \( \square \)

**Remark A.1** Let us define a new problem equivalent to (A.3) set in $\mathbb{R} \times ]-\infty, -a[$, with $F \in S(-\infty)$ as a right hand side. Then there exist $\alpha^-, \beta^- \in \mathbb{R}$ and $u^-_{res} \in S(-\infty)$ such that:

$$
u(y) = \alpha^- y_2 + \beta^- + u^-_{res}(y) \quad \text{when} \quad y_2 \to -\infty
$$

The next Lemma is easily derived from the previous one:

**Lemma A.2.** Let $F$ be a function satisfying (A.1) and (A.2), and $u$ be a solution of problem (A.3). Then:

(i) If $u \in L^2([0,1[|a, +\infty|)$, then $u$ satisfies (A.4) with $\alpha^+ = \beta^+ = 0$.

(ii) If $\frac{\partial u}{\partial y_2} \in L^2([0,1[|a, +\infty|)$, then $u$ satisfies (A.4) with $\alpha^+ = 0$.

**Remark A.2.** Again, a result of the same type can be proved in $]0,1[| - \infty, -a[.$

In the following, we define a functional space where existence and uniqueness results for the terms of the inner asymptotic expansion can be derived. We set:

$$
V = \{ u, u \text{ is } 1 - \text{periodic w.r.t. } y_1; \quad \frac{u}{\sqrt{1 + y_2^2}} \in L^2(G); \quad \nabla u \in (L^2(G))^2 \}
$$

$V$ is a linear space equipped with the following semi-norm:

$$
|u|_1 = \left( \sum_i \int_G (\frac{\partial u}{\partial y_i}(y))^2 dy \right)^{\frac{1}{2}}
$$
V is a Banach space, and the semi-norm is equivalent to the graph-norm (the proof is an extension of the one-dimensional case performed in Dautray et al\textsuperscript{15}. We can now state the following result:

**Theorem A.1.** Let $F$ be a function satisfying (A.1) and (A.2), and let $\chi \in H^{-\frac{1}{2}}(\Gamma)$ such that:

$$
\int_G Fdy + \int_\Gamma \chi ds = 0
$$

then, there exists $u \in V$ that solves:

$$
\begin{align*}
-\Delta_y u &= F \quad \text{in } G \\
\frac{\partial u}{\partial n} &= \chi \quad \text{on } \Gamma
\end{align*}
$$

(A.5)

The function $u$ is unique in $V$ (up to an additive constant), and it has the following behavior when $y_2 \rightarrow \pm \infty$:

$$
u(y) = \beta^\pm + u^\pm_{res}(y) + C
$$

for some $\beta^+$ and $\beta^-$ in $\mathbb{R}$, where $u^\pm_{res} \in S(+\infty)$, $u^-_{res} \in S(-\infty)$, and $C \in \mathbb{R}$, is an arbitrary constant.

**Proof.** The variational formulation of (A.5) is:

$$
\begin{align*}
\text{Find } u \in V \text{ such that } \\
\int_G \nabla u \nabla v dy = \int_G F v dy + \int_\Gamma \chi v ds \quad \forall v \in V
\end{align*}
$$

(A.6)

Classically, according to Lax-Milgram’s theorem, the solution $u$ exists and is unique, up to an additive constant, if $F$ and $\chi$ satisfy the first hypothesis, which is a compatibility condition. Moreover, since $u$ belongs to $V$, it satisfies:

$$
\frac{\partial u}{\partial y_k} \in L^2(G)
$$

so that $\alpha^\pm = 0$. \(\square\)

**Remark A.3** From Lemmas A.1 and A.2, it results that any function $u$ that solves (A.5) tends to $\beta^+ + C$ when $y_2 \rightarrow +\infty$, and to $\beta^- + C$ when $y_2 \rightarrow -\infty$. The difference between these two values is equal to $\beta^+ - \beta^-$, and is independant from $C$. 