An Asymptotic-Preserving Particle-In-Cell method for the Vlasov-Poisson system near quasineutrality.

Marie Hélène Vignal

IMT: Mathematics Institute of Toulouse,

University Toulouse 3

Saint-Malo the 27th january 2011

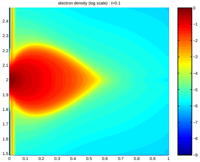
Joint works with

- P. Degond, F. Deluzet, L. Navoret, A.B. Sun

- C. Chainais-Hillairet

General topic : multiscale problems

- **Mathematic models, M_λ, depending on a parameter λ**
  - λ very small in a part of the domain
  - λ is an order 1 parameter elsewhere



- **Two-scales :** microscopic scale (small values of λ)
  and
  macroscopic scale (large values of λ)

- **Difficulties :** The classical explicit schemes are stable and consistent

  iff λ is resolved by the mesh $\Rightarrow$ huge cost

General topic : multiscale problems

A possible solution : use the asymptotic model where $\lambda \ll 1$

$$M_0 = \lim_{\lambda \to 0} M_\lambda \quad \Rightarrow \quad \text{Mesh independent of } \lambda$$

- **Problems**
  - Boundary conditions **well prepared** to the asymptotic regime
  - You must use M_λ where $\lambda = O(1) \quad \Rightarrow \quad$ **two different models**
    - Reconnection of M_λ and M_0.
    - Location of the interface.
    - Moving interface : difficult numerical problem in 2D or 3D.

General topic : multiscale problems

Another solution : use an asymptotic preserving scheme

- Use the microscopic model M_λ everywhere.
- Discretize it with a scheme which preserves the asymptotic limit $\lambda \to 0$
  - The mesh is independent of λ : **Asymptotic stability**.
  - You recover an approximate solution of M_0 when $\lambda \ll 1$:
    Asymptotic consistency.

Scheme asymptotically stable and consistent

$\Rightarrow$ Asymptotic preserving scheme (AP)

([S.Jin, 1999] kinetic $\to$ Hydro)

Particular case of quasineutral limits

- In gaz containing charged particles

$$\lambda = \frac{\text{Debye length}}{\text{size of the domain}}$$

  - Debye length is the scale of electric interactions in the gaz

- **Quasineutral limits :** $\lambda \to 0$
  - **Euler-Poisson :**
  - **Vlasov-Poisson :**
  - **Drift-diffusion Poisson :**

Some references on AP schemes in the quasineutral limit

- **Quasineutral limits**
  - **Euler-Poisson :**

    - S. Fabre (1992),
    - Ph. Colella, M.R. Dorr, D.D. Wake (1999),
    - H.H. Choe, N.S. Yoon, S.S. Kim, D.I. Choi (2001),
    - P. Crispel, P. Degond, M.H. Vignal (2007),
    - P. Degond, J.G. Liu, M.H. Vignal (2008),
    - M.H. Vignal (2010).
  - **Vlasov-Poisson :**
  - **Drift-diffusion Poisson :**

Some references on AP schemes in the quasineutral limit

- **Quasineutral limits**
  - **Euler-Poisson :**
  - **Vlasov-Poisson :**

    - R.J. Mason (1981,1983),
    - B.I. Cohen, A.B. Langdon, A. Friedman (1982,1983),
    - P. Degond, F. Deluzet, L. Navoret (2006),
    - R. Belaouar, N. Crouseilles, P. Degond, E. Sonnendrücker (2009),
    - P. Degond, F. Deluzet, L. Navoret, A.B. Sun, M.H. Vignal (2010).
  - **Drift-diffusion Poisson :**

Some references on AP schemes in the quasineutral limit

- **Quasineutral limits**
  - **Euler-Poisson :**
  - **Vlasov-Poisson :**
  - **Drift-diffusion Poisson :**

    - G. Lapenta, F. Iinoya, J.U. Brackbill (1995),
    - P.L.G. Ventzek, T.J. Sommerer, R.J. Hoekstra, M.J. Kushner, (1993),
    - P.L.G. Ventzek, R.J. Hoekstra, M.J. Kushner, (1994),
    - C. Chainais-Hillairet, M.H. Vignal.

Outline

1. General topic : multiscale problems
2. An AP scheme for the Vlasov-Poisson model in the quasineutral limit
   - The Vlasov-Poisson model and its quasineutral limit
   - The classical explicit scheme and our AP scheme
   - Numerical results
3. Generalization to the Drift-Diffusion Poisson system
   - The Drift-Diffusion Poisson system and its quasineutral limit
   - The asymptotic preserving discretizations
   - Numerical results
4. Works in progress

The Vlasov-Poisson model

- Joint work with P. Degond, F. Deluzet, L. Navoret, A.B. Sun,
- One species model for clarity

$$(VP)_\lambda \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = 0, \\ -\lambda^2 \Delta \phi = n_0 - n, \quad n = \int f \, dv. \end{cases}$$

- $f(x,v,t) =$ distribution function, $\phi(x,t) =$ electric potential,

  $n_0 =$ constant ion density, $n(x,t) =$ electron density,

$$\lambda = \frac{\text{Debye length}}{\text{size of the domain}} = \text{rescaled Debye length}$$

What is the quasineutral limit $\lambda \to 0$

- The quasineutral Vlasov system

$$(QNV) \left\{ \begin{array}{l} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = 0, \\ n = \int f \, dv = n_0 \, . \end{array} \right.$$

ϕ is the Lagrange multiplier of the constraint $n = n_0$

- Rigorous quasineutral limits : Brenier, Brenier & Grenier, Brenier & Corrias
- Explicit equation for ϕ ?

The quasineutral limit $\lambda \to 0$

- Taking the velocity moments of Vlasov and using $n = n_0$

$$\begin{cases} \partial_t n + \nabla_x \cdot (nu) = 0, \\ \partial_t (nu) + \nabla_x S = n \nabla_x \phi, \\ nu = \int f v \, dv, \quad S = \int f v \otimes v \, dv. \end{cases} \Leftrightarrow \begin{cases} \nabla_x \cdot (nu) = 0, \\ \partial_t (nu) + \nabla_x S = n \nabla_x \phi, \\ nu = \int f v \, dv, \quad S = \int f v \otimes v \, dv. \end{cases} \qquad (1)$$

- Quasineutral elliptic equation

$$\nabla_x \cdot \Big(\partial_t (nu) + \nabla_x S = n \nabla_x \phi \Big) \quad \underset{\nabla_x \cdot (nu) = 0}{\Rightarrow}$$

$$-\nabla_x \cdot (n \, \nabla_x \phi) = -\nabla_x^2 : S$$

- Reciprocally

$$-\nabla_x \cdot (n \nabla_x \phi) = -\nabla_x^2 : S \quad \Rightarrow \quad \partial_t \nabla_x \cdot (nu) = 0 \quad \Rightarrow \quad \partial_{tt}^2 n = 0$$

The quasineutral limit $\lambda \to 0$

- Reformulated quasineutral Vlasov system

$$(RQNV) \begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f = 0, \\ -\nabla_x \cdot (n \, \nabla_x \phi) = -\nabla_x^2 : S, \\ (n = n_0)|_{t=0} \text{ and } \dfrac{d}{dt}(n = n_0)|_{t=0}, \\ n = \displaystyle\int f \, dv, \qquad S = \int f \, v \otimes v \, dv. \end{cases}$$

- Equivalent to the quasineutral Vlasov model

The reformulated Vlasov-Poisson system

- Is it possible to complete the diagram ?

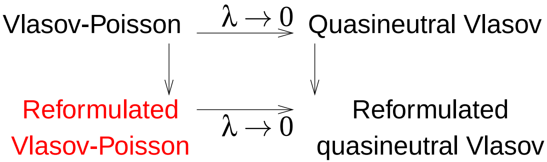
Vlasov-Poisson $\xrightarrow{\lambda \to 0}$ Quasineutral Vlasov

$\downarrow$

Reformulated
quasineutral Vlasov

The reformulated Vlasov-Poisson system

- It is possible to complete the diagram.



- Starting from Vlasov-Poisson and taking the velocity moments

$$\begin{cases} \partial_t n + \nabla_x \cdot (n u) = 0, & (2) \\ \partial_t (n u) + \nabla_x S = n \nabla_x \phi. & (3) \end{cases}$$

- $\nabla_x \cdot (3) - \partial_t (2) \quad \Rightarrow \quad -\partial_{tt}^2 n - \nabla_x \cdot (n\, \nabla_x \phi) = -\nabla_x^2 : S$

- $n = n_0 + \lambda^2 \Delta \phi \quad \Rightarrow$

The reformulated Poisson equation

$$\lambda^2 \partial_{tt}^2 (-\Delta \phi) - \nabla_x \cdot (n\, \nabla_x \phi) = -\nabla_x^2 : S$$

The reformulated Vlasov-Poisson system

$$(RVP)\begin{cases} \partial_t f + v\cdot\nabla_x f + \nabla_x\phi\cdot\nabla_v f = 0, \\ \lambda^2\partial_{tt}^2(-\Delta\phi) - \nabla_x\cdot(n\nabla_x\phi) = -\nabla_x^2 : S, \qquad (4) \\ (\lambda^2\Delta\phi = n - n_0)|_{t=0} \text{ and } \frac{d}{dt}(\lambda^2\Delta\phi = n - n_0)|_{t=0}. \end{cases}$$

- Equivalent to the Vlasov-Poisson model.
- Does not degenerate when $\lambda \to 0$.
- Reduces to (RQN) system when $\lambda = 0$.

} Consistency with $\lambda \to 0$

- (4) is a harmonic oscillator Eq. on $-\Delta\phi$
- Oscillations in time at the period λ
  - Explicit scheme $\Rightarrow$ conditional stab. $\Delta t \leq \lambda$
  - Implicit scheme $\Rightarrow$ unconditional stability

Outline

1 General topic : multiscale problems

2 An AP scheme for the Vlasov-Poisson model in the quasineutral limit
- The Vlasov-Poisson model and its quasineutral limit
- The classical explicit scheme and our AP scheme
- Numerical results

3 Generalization to the Drift-Diffusion Poisson system
- The Drift-Diffusion Poisson system and its quasineutral limit
- The asymptotic preserving discretizations
- Numerical results

4 Works in progress

The classical scheme, semi-discretization in time

- If f^m approximates f at time t^m $\quad (n^m = \int f^m\,dv, \quad (nu)^m = \int f^m\, v\, dv)$

$$\begin{cases} \dfrac{f^{m+1} - f^m}{\Delta t} + v \cdot \nabla_x f^m + \nabla_x \phi^{m+1} \cdot \nabla_v f^m = 0, \\ -\lambda^2 \Delta \phi^{m+1} = n_0 - n^{m+1}. \end{cases}$$

- Uncoupled scheme :
  - Integrating Vlasov eq. $\Rightarrow \dfrac{n^{m+1} - n^m}{\Delta t} + \nabla_x \cdot (nu)^m = 0$ gives n^{m+1} ,
  - Poisson gives ϕ^{m+1},
  - Vlasov gives f^{m+1}.

The discret reformulated Poisson equation

$$-\lambda^2 \frac{\Delta\phi^{m+1} - 2\,\Delta\phi^m + \Delta\phi^{m-1}}{\Delta t^2} - \nabla_x \cdot \left(n^{m-1}\, \nabla_x \phi^m\right) = \ldots$$

- Stable and consistent iff $\Delta t \leq \lambda$.

The AP scheme, semi-discretization in time

- If f^m approximates f at time t^m $\quad (n^m = \int f^m \, dv, \quad (nu)^m = \int f^m v \, dv)$

$$\begin{cases} \dfrac{f^{m+1} - f^m}{\Delta t} + v \cdot \nabla_x f^{m+1} + \nabla_x \phi^{m+1} \cdot \nabla_v f^m = 0, \\ -\lambda^2 \Delta \phi^{m+1} = n_0 - \tilde{n}^{m+1}. \end{cases}$$

- Uncoupled scheme ?
  - Integrating Vlasov eq. $\Rightarrow$

$$\frac{n^{m+1} - n^m}{\Delta t} + \nabla_x \cdot (nu)^{m+1} = 0,$$

$$\frac{(nu)^{m+1} - (nu)^m \, dv}{\Delta t} + \nabla_x \int f^{m+1} v \otimes v \, dv = n^m \nabla_x \phi^{m+1},$$

$$n^{m+1} = n^m - \Delta t \, \nabla_x \cdot (nu)^m + \Delta t^2 \left(\nabla_x^2 : \int f^{m+1} v \otimes v \, dv - \nabla_x \cdot \left(n^m \nabla_x \phi^{m+1} \right) \right)$$

The AP scheme, semi-discretization in time

- If f^m approximates f at time t^m $\quad (n^m = \int f^m\,dv, \quad (nu)^m = \int f^m\,v\,dv)$

$$\begin{cases} \dfrac{f^{m+1}-f^m}{\Delta t} + v\cdot\nabla_x f^{m+1} + \nabla_x\phi^{m+1}\cdot\nabla_v f^m = 0, \\ -\lambda^2\Delta\phi^{m+1} = n_0 - \tilde{n}^{m+1}. \end{cases}$$

- Uncoupled scheme
  - Replace n^{m+1} by $\tilde{n}^{m+1}$ where

$$\tilde{n}^{m+1} = n^m - \Delta t\,\nabla_x\cdot(nu)^m + \Delta t^2\left(\nabla_x^2 : \int f^m\,v\otimes v\,dv - \nabla_x\cdot\left(n^m\,\nabla_x\phi^{m+1}\right)\right)$$

- Unconditionally stable and consistent ~~$\Delta t \leq \lambda$~~

Outline

1 General topic : multiscale problems

2 An AP scheme for the Vlasov-Poisson model in the quasineutral limit
- The Vlasov-Poisson model and its quasineutral limit
- The classical explicit scheme and our AP scheme
- Numerical results

3 Generalization to the Drift-Diffusion Poisson system
- The Drift-Diffusion Poisson system and its quasineutral limit
- The asymptotic preserving discretizations
- Numerical results

4 Works in progress

The test case : Grismayer, Mora, Adam, Héron, Phys. Rev. 2008

- One dimensional two-species plasma expansion test case
  - Vlasov for ions $$\partial_t f_i + v\,\partial_x f_i - \partial_x\phi\,\partial_v f_i = 0,$$
  - Vlasov for electrons $$\partial_t f_e + v\,\partial_x f_e + \frac{\partial_x\phi}{\varepsilon}\,\partial_v f_e = 0,$$
  - Poisson $$-\lambda^2\partial_{xx}^2\phi = n_i - n_e, \quad n_{i,e} = \int f_{i,e}\,dv.$$

$$\varepsilon = \frac{\text{electron mass}}{\text{ion mass}}, \quad \text{Debye length} = \lambda \ll 1, \quad \text{Plasma period} = \sqrt{\varepsilon}\lambda \ll 1.$$

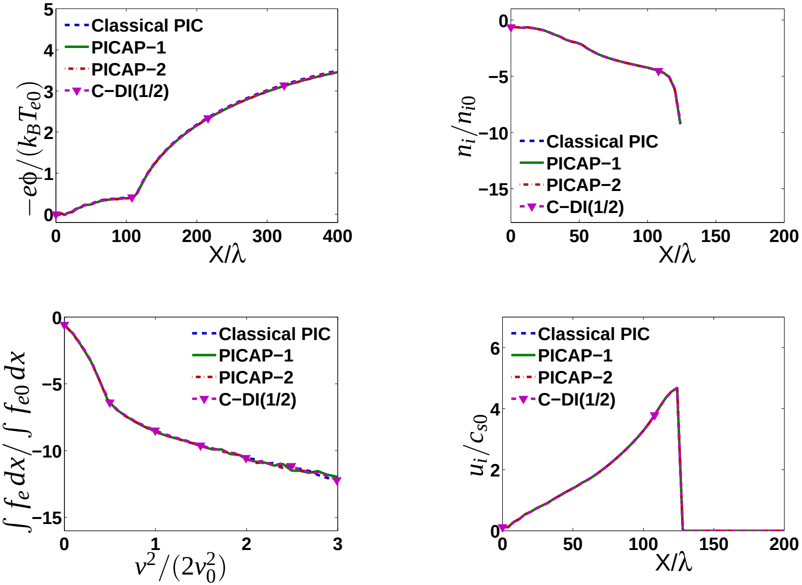
- Initially, ions and electrons are Maxwellian
  - Ion slab $$n_{i0} = \begin{cases} 1, & 0 \le x \le 20\lambda, \\ 0, & 20\lambda \le x \le 3.10^4. \end{cases}$$
  - Electrons in a Boltzmanian equilibrium $$\begin{cases} n_{e0} = \exp(\phi_0), \\ -\partial_{xx}^2\phi_0 = n_{i0} - \exp(\phi_0). \end{cases}$$

The test case

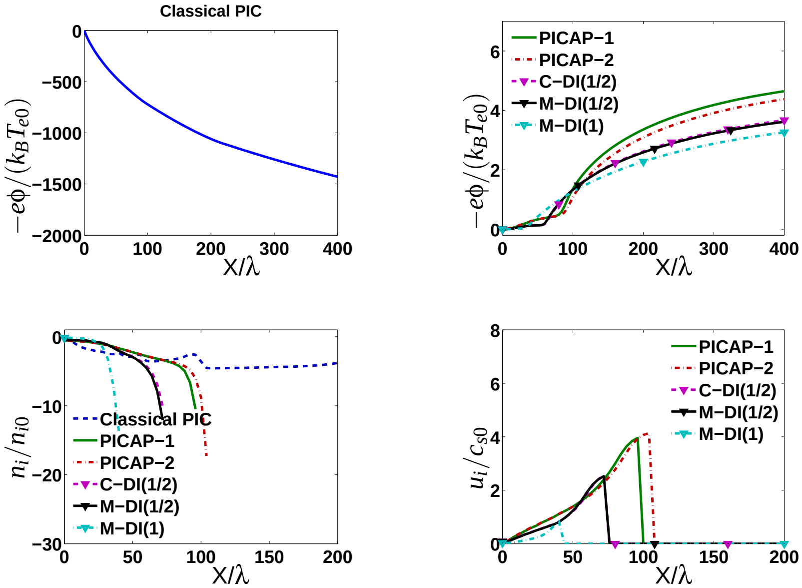
- Numerical parameters :
  - Domain : $x \in [0, 3.10^4 \lambda]$, with $\Delta x = 4\lambda \quad \Rightarrow \quad 7500$ cells.
  - Total number of numerical particles (ions+electrons) $\approx 5.10^6$.
  - Resolved case : $\Delta t = 0.05\sqrt{\varepsilon}\lambda$, Unresolved case : $\Delta t = 3\sqrt{\varepsilon}\lambda$

- Comparison between :
  - Explicit PIC scheme $\Rightarrow$ conditional stability $\Delta t \leq \sqrt{\varepsilon}\lambda$
  - Our AP-PIC scheme $\Rightarrow$ unconditional stability ~~$\Delta t \leq \sqrt{\varepsilon}\lambda$~~
  - Direct-Implicit PIC schemes : Cohen, Friedman, Langdon $\Rightarrow$ ~~$\Delta t \leq \sqrt{\varepsilon}\lambda$~~ (modified on the 2 first steps)

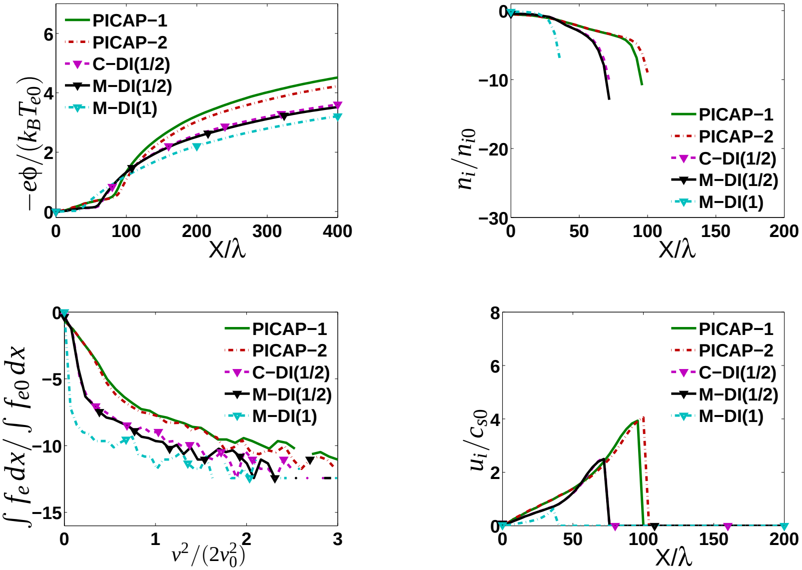
Resolved case : $\Delta t = 0.05\sqrt{\varepsilon}\lambda, \quad \Delta x = 4\lambda$



Unresolved case : $\Delta t = 3\sqrt{\varepsilon}\lambda, \quad \Delta x = 4\lambda$



Unresolved case, less particles $/20$



Gain of CPU time

- Ratios

$$\frac{\text{CPU Resolved case}}{\text{CPU Unresolved case}} = 48,$$

$$\frac{\text{CPU Resolved case}}{\text{CPU Unresolved case less particles}} = 960.$$

- About 1000 times faster in one dimension.

Outline

1. General topic : multiscale problems
2. An AP scheme for the Vlasov-Poisson model in the quasineutral limit
   - The Vlasov-Poisson model and its quasineutral limit
   - The classical explicit scheme and our AP scheme
   - Numerical results
3. Generalization to the Drift-Diffusion Poisson system
   - The Drift-Diffusion Poisson system and its quasineutral limit
   - The asymptotic preserving discretizations
   - Numerical results
4. Works in progress

The scaled Drift-Diffusion Poisson system

- Joint work with C. Chainais-Hillairet (University Lille 1)

$$(DD)_\lambda \begin{cases} \partial_t N - \nabla \cdot \Big(\nabla r(N) - N \nabla \phi \Big) = 0, \\ \partial_t P - \nabla \cdot \Big(\nabla r(P) + P \nabla \phi \Big) = 0, \\ -\lambda^2 \Delta \phi = P - N + C, \end{cases} \qquad r(s) = s^\gamma, \quad \gamma \geq 1.$$

- $N =$ electron density, $P =$ hole density,
  $\phi =$ electric potential, $C(x) =$ doping profile,

$\lambda =$ rescaled Debye length

- Existence, sometimes uniqueness :
  -T. Seidman, G. Troianello (1985), -H. Gajewski (1985),
  -P.A. Markowitch, C.A. Ringhofer, C. Schmeiser (1990).

- Proof based on estimates in L^∞, $L^2(0,T;H^1(\Omega))$ and Schauder fixed point

The formal quasineutral limit $\lambda \to 0$:

$$(DD)_0 \begin{cases} \partial_t N - \nabla \cdot \Big(\nabla r(N) - N \nabla \phi \Big) = 0, \\ \partial_t P - \nabla \cdot \Big(\nabla r(P) + P \nabla \phi \Big) = 0, \\ 0 = P - N + C. \end{cases}$$

- Substracting mass eqs.

$$\partial_t (P - N) - \Delta(r(P) - r(N)) - \nabla \cdot \Big((P + N) \nabla \phi \Big) = 0.$$

- Using the quasineutrality constraint $\Rightarrow$ explicit eq. for the potential

$$-\nabla \cdot \Big((P + N) \nabla \phi \Big) = \Delta(r(P) - r(N)).$$

- Starting from the drift-diffusion Poisson model

  Same transformations $\Rightarrow$ The reformulated Poisson equation

$$-\lambda^2 \partial_t \Delta \phi - \nabla \cdot \Big((P + N) \nabla \phi \Big) = \Delta(r(P) - r(N)).$$

  - Solutions oscillate at the period λ^2.
  - Explicit schemes must resolve this scale for stability $\Rightarrow \Delta t \leq \lambda^2$.

Rigorous quasineutral limit

- I. Gasser, C.D. Levermore, P.A. Markowich, C. Schmeiser (2001)

$$(DD)_\lambda \begin{cases} \partial_t N - \nabla \cdot \Big(\nabla N - N \nabla \phi \Big) = 0, \\ \partial_t P - \nabla \cdot \Big(\nabla P + P \nabla \phi \Big) = 0, \\ -\lambda^2 \Delta \phi = P - N + C, \end{cases} \qquad x \in \Omega \subset \mathbb{R}^d, t > 0.$$

- Mixed quasineutral boundary conditions (Dirichlet and Neumann)

- On $\partial\Omega_D$: $N = N_b, \quad P = P_b, \quad \phi = \phi_b,$ with $P_b - N_b + C = 0.$

- On $\partial\Omega_N$: $(\nabla N - N\nabla\phi) \cdot \nu = 0, \quad (\nabla P + P\nabla\phi) \cdot \nu = 0, \quad \nabla\phi \cdot \nu = 0.$

- Quasineutral initial conditions

$$P_0 - N_0 + C = 0.$$

The rigorous quasineutral limit

Theorem : (Gasser-Levermore-Markowich-Schmeiser)

If - $C \in H^1(\Omega)$, - P_0, $N_0 \in L^1(\Omega)$ uniformaly in λ,

- $N + P \geq \underline{C}$, - Entropy (energy) $e(t = 0)$ is independant of λ.

Then $N^\lambda, P^\lambda \to N, P$ in $L^1([0,T] \times \Omega)$ and $\nabla\phi^\lambda \rightharpoonup \nabla\phi$ in $L^2(\Omega \times [0,T])$,

$$(DD)_0 \begin{cases} \partial_t N - \nabla \cdot \Big(\nabla N - N\,\nabla\phi\Big) = 0, \\ \partial_t P - \nabla \cdot \Big(\nabla P + P\,\nabla\phi\Big) = 0, \\ 0 = P - N + C. \end{cases}$$

- On $\partial\Omega_D$: $N = N_b, \quad P = P_b, \quad \phi = \phi_b.$
- On $\partial\Omega_N$: $(\nabla N - N\nabla\phi) \cdot \nu = 0, \quad (\nabla P + P\nabla\phi) \cdot \nu = 0.$

$$P_0 - N_0 + C = 0.$$

The rigorous quasineutral limit

- Choosing $P+N$, $P-N$ and ϕ for unknowns

$$(DD)_0 \Leftrightarrow \begin{cases} \partial_t (P+N) - \nabla \cdot \Big(\nabla (P+N) - C\,\nabla \phi \Big) = 0, \\ -\nabla \cdot \Big(-\nabla C + (P+N)\,\nabla \phi \Big) = 0, \\ P - N = -C. \end{cases}$$

- On $\partial\Omega_D$: $\quad P+N = P_b + N_b, \quad P-N = P_b - N_b = -C, \quad \phi = \phi_b.$
- On $\partial\Omega_N$: $\quad (\nabla(P+N) - C\,\nabla\phi)\cdot\nu = 0, \quad (-\nabla C + (P+N)\nabla\phi)\cdot\nu = 0.$

- Proof based on :
  - the entropy dissipation,
  - Sobolev imbeddings.

- If $C = 0$, simpler, because estimates in L^∞ and $L^2(0,T;H^1(\Omega))$ are uniform in λ, (I. Gasser, 2001).

Outline

1. General topic : multiscale problems
2. An AP scheme for the Vlasov-Poisson model in the quasineutral limit
   - The Vlasov-Poisson model and its quasineutral limit
   - The classical explicit scheme and our AP scheme
   - Numerical results
3. Generalization to the Drift-Diffusion Poisson system
   - The Drift-Diffusion Poisson system and its quasineutral limit
   - The asymptotic preserving discretizations
   - Numerical results
4. Works in progress

The implicit scheme

- If N^n, P^n, ϕ^n are known approximations at time t^n

$$\left\{ \begin{array}{l} N^{n+1} - N^n - \Delta t\, \nabla \cdot \left(\nabla N^{n+1} - N^{n+1} \nabla \phi^{n+1} \right) = 0, \\ P^{n+1} - P^n - \Delta t\, \nabla \cdot \left(\nabla P^{n+1} + P^{n+1} \nabla \phi^{n+1} \right) = 0, \\ -\lambda^2 \Delta \phi^{n+1} = P^{n+1} - N^{n+1} + C. \end{array} \right.$$

- Coupled scheme $\Rightarrow$ non linear $\Rightarrow$ Newton algorithm.

- Unconditionally stable scheme $\Rightarrow$ asymptotic preserving scheme.

The implicit scheme, rigorous results

- Existence and uniqueness results
  - Semi-discretization in time : - estimates in L^∞, $L^2(0,T;H^1(\Omega))$,
    - Schauder fixed point.
  - Scheme in space and time : - L^∞ estimates, topological degree,
    (C. Chainais-Hillairet, J.G. Liu, Y.J. Peng, 2003,2004).

- For $C=0$, estimates in L^∞ and $L^2(]0,T[;H^1(\Omega))$ are uniform in λ.

- Discrete dissipation of the entropy (Marianne Chatard, to appear).

- For the semi-discretization in time : result similar to the convergence proof of Gasser-Levermore-Markowich-Schmeiser can be proved.

Our uncoupled asymptotic preserving scheme

- A scheme stable uniformaly
  - Implicit coupling terms $\Rightarrow$ stability uniform in λ

$$\begin{cases} N^{n+1} - N^n - \Delta t \, \nabla \cdot \left(\nabla N^n - N^n \, \nabla \phi^{n+1} \right) = 0, & (1) \\ P^{n+1} - P^n - \Delta t \, \nabla \cdot \left(\nabla P^n + P^n \, \nabla \phi^{n+1} \right) = 0, & (2) \\ -\lambda^2 \, \Delta \phi^{n+1} = P^{n+1} - N^{n+1} + C, & (3) \end{cases}$$

  - Eliminating P^{n+1} and N^{n+1}, equation (3) yields

$$-\lambda^2 \, \Delta \phi^{n+1} - \Delta t \, \nabla \cdot \left((N^n + P^n) \, \nabla \phi^{n+1} \right) = P^n - N^n + C + \Delta t \; \Delta (P^n - N^n).$$

  - Discret reformulated Poisson equation

$$-\nabla \cdot \left(\left(\lambda^2 + \Delta t \, (N^n + P^n) \right) \nabla \phi^{n+1} \right) = P^n - N^n + C + \Delta t \; \Delta (P^n - N^n).$$

Our uncoupled asymptotic preserving scheme

➠ We replace in the implicit scheme the Poisson eq. by the reformulated Poisson eq.

$$\left\{\begin{array}{l} N^{n+1} - N^n - \Delta t\, \nabla \cdot \left(\nabla N^{n+1} - N^{n+1}\, \nabla \phi^{n+1} \right) = 0, \\ P^{n+1} - P^n - \Delta t\, \nabla \cdot \left(\nabla P^{n+1} + P^{n+1}\, \nabla \phi^{n+1} \right) = 0, \\ -\nabla \cdot \left(\left(\lambda^2 + \Delta t\, (N^n + P^n) \right) \nabla \phi^{n+1} \right) = P^n - N^n + C + \Delta t\; \Delta(P^n - N^n). \end{array}\right.$$

- ➠ Uncoupled scheme $\Rightarrow$ linear.
- ➠ Asymptotic preserving scheme (verified in numerical results).

Our uncoupled asymptotic preserving scheme, results

- Existence and uniqueness of the solution : like for the implicit scheme

- Formally
  - $\lambda = 0$ in the scheme

$$\begin{cases} N^{n+1} - N^n - \Delta t\, \nabla \cdot \Big(\nabla N^{n+1} - N^{n+1} \nabla \phi^{n+1} \Big) = 0, \\ P^{n+1} - P^n - \Delta t\, \nabla \cdot \Big(\nabla P^{n+1} + P^{n+1} \nabla \phi^{n+1} \Big) = 0, \\ -\nabla \cdot \Big(\Big(\Delta t\, (N^n + P^n) \Big) \nabla \phi^{n+1} \Big) = P^n - N^n + C + \Delta t\, \Delta (P^n - N^n). \end{cases}$$

  - Asymptotic expansion in Δt

$\Delta t^0 \quad \rightarrow \quad$ The quasineutrality : $P^n - N^n + C = 0$,

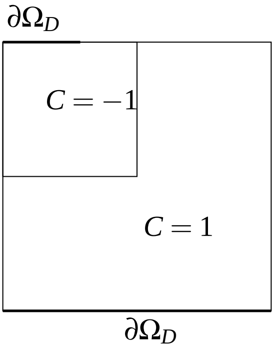
$\Delta t^1 \quad \rightarrow \quad$ discretization of the quasineutral eq. for the potential ϕ :

$$-\nabla \cdot \Big(\Big((N^n + P^n) \Big) \nabla \phi^{n+1} \Big) = \Delta (P^n - N^n) = -\Delta C.$$

Outline

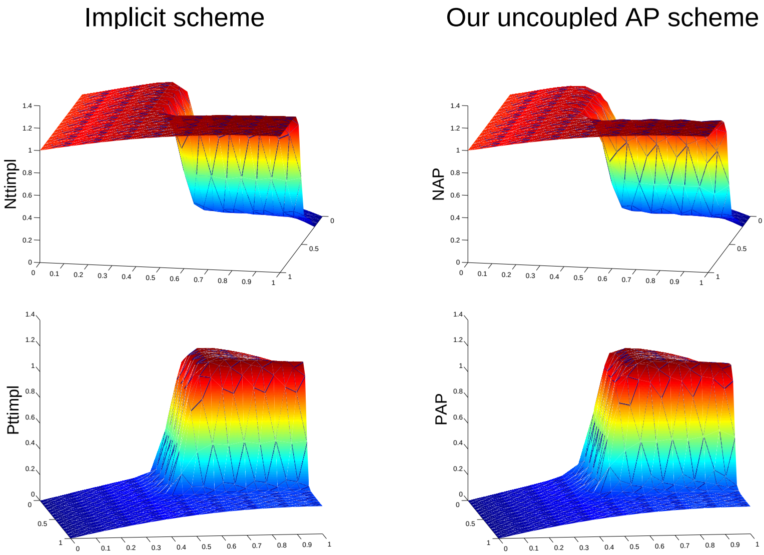
1. General topic : multiscale problems
2. An AP scheme for the Vlasov-Poisson model in the quasineutral limit
   - The Vlasov-Poisson model and its quasineutral limit
   - The classical explicit scheme and our AP scheme
   - Numerical results
3. Generalization to the Drift-Diffusion Poisson system
   - The Drift-Diffusion Poisson system and its quasineutral limit
   - The asymptotic preserving discretizations
   - Numerical results
4. Works in progress

PN Diode



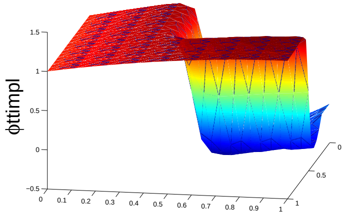
- $\lambda^2 = 10^{-3}$ or 10^{-10}, $\Delta t = 5\ 10^{-4}$.
- Final time T=0.5
- Mesh in space : 896 triangles.
- Boundary conditions :
  - On $\partial\Omega_D$: quasineutral Dirichlet conditions.
  - Elsewhere : homogeneous Neumann conditions.
- Quasineutral initial conditions.

PN Diode, $\lambda^2 = 10^{-3}$, $\Delta t = 5\ 10^{-4}$

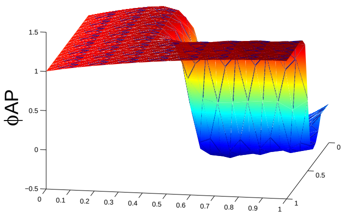


PN Diode, $\lambda^2 = 10^{-3}$, $\Delta t = 5\ 10^{-4}$

Implicit scheme



Our uncoupled AP scheme



-CPU time of the implicit scheme : 86.7656s.

-CPU time of our uncoupled AP scheme : 16.5000s.

$\Rightarrow$ ratio ≈ 5.3.

-relative errors in L^2 norms : On N : 0.0532,

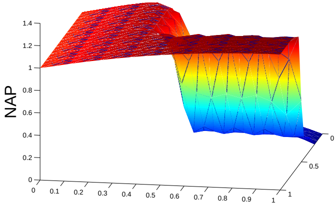
On P : 0.0482,

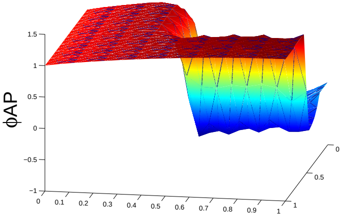
On ϕ : 0.0667.

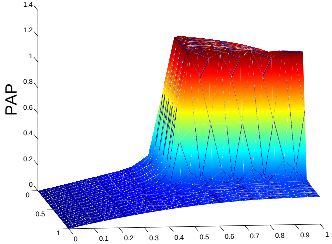
PN Diode, $\lambda^2 = 10^{-10}$, $\Delta t = 5\ 10^{-4}$

Implicit scheme is unstable

Our uncoupled AP scheme







Outline

1. General topic : multiscale problems
2. An AP scheme for the Vlasov-Poisson model in the quasineutral limit
   - The Vlasov-Poisson model and its quasineutral limit
   - The classical explicit scheme and our AP scheme
   - Numerical results
3. Generalization to the Drift-Diffusion Poisson system
   - The Drift-Diffusion Poisson system and its quasineutral limit
   - The asymptotic preserving discretizations
   - Numerical results
4. Works in progress

Works in progress

- Uniform convergence of implicit scheme (full discretization) and of our AP scheme.
- Study of the numerical initial and boundary layers.
- Long time behavior of the scheme
  Semi-implicit scheme (C. Chainais-Hillairet, F. Filbet, 2010).
- Corrosion model of iron (C. Chainais-Hillairet, C. Bataillon)

$$\begin{cases} \varepsilon \partial_t N - \nabla \cdot \Big(\nabla N - N \nabla \phi \Big) = 0, \\ \partial_t P - \nabla \cdot \Big(\nabla P + P \nabla \phi \Big) = 0, \\ -\lambda^2 \Delta \phi = P - N + C. \end{cases}$$

Two small parameters ε and λ, build an AP scheme for the double limit.